Traveling Wave Undulators for FELs and Synchrotron Radiation Sources

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1. Introduction

We study the use of a traveling wave waveguide as an undulator for short wavelength free-electron lasers (FELs) and synchrotron radiation sources. This type of undulator—which we will call TWU—can be useful when a short electron oscillation period and a large aperture for the propagation of the beam are needed. The availability of high power X-band microwave sources, developed for the electron-positron linear collider, make it possible today to build TWUs of practical interest to produce short wavelength radiation from a beam of reduced energy respect to the case of more conventional undulators. In this paper we will discuss the characteristic of the TWU, the systems that can be used to control the effects of RF power losses in the waveguide walls, and how to optimize a TWU and the associated electron transport system for use in a synchrotron radiation source or FEL.

Microwave undulators have been considered before in a standing wave configuration [1], [2]. Measurements of the spontaneous undulator radiation when propagating a beam with energy of 143 or 220 MeV through the undulator have also been reported in reference[1]. A discussion of the parameters of radio frequency undulator in various configurations has also been made recently in reference [3]. The use of a radio frequency wiggler operating at 30 GHz, as a damping device in a damping ring for a linear collider has been considered in reference [4].

In this paper we discuss a traveling wave, radio frequency undulator operating at 12 Ghz. The reason for the choice of this frequency is that high power RF sources operating near this frequency, and pulse compression techniques have been developed as part of the international linear collider R&D [5]. Using these sources and pulse compression, power level as high as 450MW are now available, with pulse duration of hundreds of nanoseconds. This power level makes possible to consider a traveling wave, radio frequency undulator (TWU) as a a practical and interesting part of linac based synchrotron radiation sources and free-electron lasers. As we will discuss in the following undulator parameter values of the order of 0.4 for an undulator period of 1.45 cm are possible. The corresponding waveguide transverse size is about 1.8 cm.

Some of the advantages of a TWU are: short undulator period; the possibility of

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1 This work was performed in support of the LCLS project at SLAC.
producing a helical electron trajectory using two modes in the TWU as will be discussed in the next section; the possibility of changing the electron trajectory from a helical to a linear oscillation, thus changing the polarization of the undulator radiation from circular to linear, including the intermediate elliptical polarization; the possibility of controlling the polarization for each electron and radiation pulse; the low level of background synchrotron radiation because of the low value of the undulator parameter and of the helical polarization, which reduces the contribution of high harmonics on axis. Another important advantage, when the TWU is compared with a conventional magnetic undulator, is the large transverse aperture of waveguide, about 1.8 cm. The conventional undulator would have a gap of about 6 mm for the same values of the period and undulator parameter.

The helical polarization, the large aperture, and the reduced spontaneous radiation power level also make the TWU an interesting future option for X-ray FELs, like LCLS and Tesla, which will use many different specialized undulators.

For a practical use of the undulator some of the problems that will need to be addressed are: reproducibility of the radio frequency power from pulse to pulse; variation in the wave amplitude and phase during the pulse. The first effect would produce a fluctuation in the undulator radiation central wavelength; the second can lead to a broadening of the spontaneous radiation spectrum and a reduction in gain for the free-electron laser case. These effects are not discussed in this paper.

In the following sections we will first evaluate the field in the TWU and the electron trajectory for the case when we propagate through a rectangular waveguide two modes, m=0, n=1 and m=1, n=0, shifted in phase by \( \pi / 2 \). The electron trajectory for this configuration is a helix with a period near to one half of the vacuum radiofrequency wavelength. We discuss also the defocusing effect of the radio frequency forces acting on the electron. We the consider the use of the TWU to generate spontaneous undulator radiation, and for a short wavelength free-electron laser. We show that a 1 GeV linac and the TWU can be used to design a water window laser, covering the 2 to nm wavelength. In the last section we discuss the effect of the radio frequency field power losses in the metallic walls of the waveguide. The loss gives a reduction of the undulator parameter and of the undulator radiation wavelength along the length of the TWU, thus broadening the spectrum and reducing the free-electron laser gain. We show that a small tapering of the waveguide transverse size can be used to counteract the power losses, and avoid the gain reduction.

2. Fields in the waveguide

Consider a cylindrical waveguide with a rectangular cross section of sides a, b. Following Jackson \(^6\), the field in the traveling wave in the waveguide can be written as a superposition of modes characterized by two numbers, m and n. If we consider transverse electric modes, and we use a reference frame with the z-axis along one of the TWU sides, the fields are given for each mode by:

\[
H_z = H_0 \cos\left(\frac{\pi m x}{a}\right) \cos\left(\frac{\pi n y}{b}\right) \exp\left[i (k_{m,n} z - \omega_{rf} t)\right],
\] 

(1)
\[ E_z = 0, \]  
\[ E_x = -i H_0 \mu_0 \omega_{rf} \frac{\pi n / b}{(\pi m / a)^2 + (\pi n / b)^2} \cos \left( \frac{\pi m x}{a} \right) \sin \left( \frac{\pi n y}{b} \right) \exp \left[ i (k_{m,n} z - \omega_{rf} t) \right], \]  
\[ E_y = i H_0 \mu_0 \omega_{rf} \frac{\pi m / a}{(\pi m / a)^2 + (\pi n / b)^2} \sin \left( \frac{\pi m x}{a} \right) \cos \left( \frac{\pi n y}{b} \right) \exp \left[ i (k_{m,n} z - \omega_{rf} t) \right], \]  
\[ B_x = -\frac{k_{m,n}}{\omega_{rf}} E_y, \]  
\[ B_y = \frac{k_{m,n}}{\omega_{rf}} E_x, \]

where

\[ k_{m,n} = \pm \sqrt{\left( \frac{\omega_{rf}}{c} \right)^2 - \left( \frac{\Omega_{m,n}}{c} \right)^2}, \]

and the cutoff frequency \( \Omega_{m,n} \) is given by

\[ \Omega_{m,n} = c \sqrt{\left( \frac{\pi m}{a} \right)^2 + \left( \frac{\pi n}{b} \right)^2}. \]

The plus sign in (7) is for a wave co-propagating with the electron beam, and the minus sign for the opposite case. We assume for the TWU that the wave and the electrons are counter-propagating, and choose the minus sign.

The power flow in the waveguide for one given mode is given by

\[ P = \frac{ab}{8} Z_0 H_0^2 \left( \frac{\omega_{rf}}{\Omega_{m,n}} \right)^2 \sqrt{1 - \left( \frac{\Omega_{m,n}}{\omega_{rf}} \right)^2}, \]

where \( Z_0 = \sqrt{\mu_0 / \varepsilon} = 120\pi \Omega \) is the vacuum impedance.

The group velocity of the field in the waveguide is

\[ V_G = c \left[ 1 - \left( \frac{\Omega_{m,n}}{\omega_{rf}} \right)^2 \right]. \]

In what follows we will consider only two modes \( m=0, n=1 \) and \( m=1, n=0 \), shifted in phase by \( \pi / 2 \). We will also shift to new coordinates \((\xi, \zeta)\), such that
\[ x = \frac{a}{2} + \xi, \quad y = \frac{b}{2} + \zeta \]  
\[ \text{(11)} \]

For these two modes the fields are

\[ E_{x,0,1} = -i \frac{\omega_0 a}{\pi} \mu H_0 \cos \left( \frac{\pi \xi}{a} \right) \exp \left[ i ( k_{0,1} z - \omega_\sigma t) \right] \]  
\[ \text{(12)} \]

\[ E_{y,0,1} = 0 \]  
\[ \text{(13)} \]

\[ H_{z,0,1} = -H_0 \sin \left( \frac{\pi \xi}{a} \right) \exp \left[ i ( k_{0,1} z - \omega_\sigma t) \right] \]  
\[ \text{(14)} \]

\[ E_{x,1,0} = 0 \]  
\[ \text{(15)} \]

\[ E_{y,1,0} = i \frac{\omega_0 a}{\pi} \mu H_0 \cos \left( \frac{\pi \xi}{a} \right) \exp \left[ i ( k_{0,1} z - \omega_\sigma t) + \pi/2 \right] \]  
\[ \text{(16)} \]

\[ H_{z,1,0} = -H_0 \sin \left( \frac{\pi \xi}{a} \right) \exp \left[ i ( k_{0,1} z - \omega_\sigma t) + \pi/2 \right] \]  
\[ \text{(17)} \]

where \( k_{0,1} = k_{1,0} = \pm \sqrt{\left( \frac{\omega_\sigma}{c} \right)^2 - \left( \frac{\pi}{a} \right)^2} \).

The corresponding transverse magnetic fields can be obtained from (5), (6).

The power in the waveguide decreases exponentially because of energy losses in the waveguide metallic walls. The attenuation length, reducing the power to 1/e of the initial value, is given, for the case of a (1,0) or (0,1) mode, by

\[ \beta_{1,0} = \frac{4}{a Z_0 \sigma \delta(\omega_\sigma)} \sqrt{\frac{\omega_\sigma a}{\pi c}} \left( \frac{1}{2} + \left( \frac{\pi c}{\omega_\sigma a} \right)^2 \right), \]  
\[ \text{(18)} \]

where \( \sigma \) and \( \delta(\omega_\sigma) \) are the metal conductivity and skin depth evaluated at the cutoff frequency

\[ \delta = \sqrt{\frac{2a}{\pi Z_0 \sigma}}. \]  
\[ \text{(19)} \]

3. Forces

We now assume that the waveguide has a square cross section, \( a = b \), and that
two modes, $m=1$, $n=0$ and $m=0$, $n=1$ propagate through the waveguide. We also assume that the mode $m=1$, $n=0$ is shifted by $\pi/2$ respect to the other one. We further introduce an additional transverse focusing force, as it could be produced by a system of quadrupoles along the TWU. We evaluate the force acting on one electron moving near to the waveguide axis with transverse coordinates $x = \frac{a}{2} + \xi$, $y = \frac{b}{2} + \zeta$. The result is

$$F_x = \frac{-a}{\pi} \cos\left(\frac{\pi \xi}{a}\right)(\omega_{rf} - \beta_z c k_{0,1}) \sin(\omega_{rf} t - k_{0,1} z) - \beta_x c \left(\sin\left(\frac{\pi \xi}{a}\right) \cos(\omega_{rf} t - k_{0,1} z) + \sin\left(\frac{\pi \zeta}{a}\right) \sin(\omega_{rf} t - k_{0,1} z)\right) - \Omega^2 m \gamma \xi$$

(20)

$$F_y = \frac{-a}{\pi} \cos\left(\frac{\pi \zeta}{a}\right)(\omega_{rf} - \beta_z c k_{0,1}) \cos(\omega_{rf} t - k_{0,1} z) + \beta_y c \left(\sin\left(\frac{\pi \xi}{a}\right) \cos(\omega_{rf} t - k_{0,1} z) + \sin\left(\frac{\pi \zeta}{a}\right) \sin(\omega_{rf} t - k_{0,1} z)\right) - \Omega^2 m \gamma \zeta$$

(21)

$$F_z = \frac{-k_{0,1} a}{\pi} \left\{ \beta_x c \cos\left(\frac{\pi \xi}{a}\right) \sin(\omega_{rf} t - k_{0,1} z) + \beta_y c \cos\left(\frac{\pi \zeta}{a}\right) \cos(\omega_{rf} t - k_{0,1} z) \right\}.$$  

(22)

where $\beta_x$, $\beta_y$, $\beta_z$, are the components of the electron velocity in units of $c$.

Notice that in the second terms in the transverse forces are proportional to the particle and velocity and the displacement from axis, and are nonlinear terms. Notice also that for the first term the strength of the force, like the transverse components of the field, decreases as one moves off the TWU axis. This is the opposite situation to that of a static magnetic undulator, where the field is minimum on axis and increases as we move off axis.

4. Equations of motion

The forces given in (20)-(22) give a complicated set of equations of motion. We solve them with an approximation procedure. We first neglect the non linear terms, and assume that the dominant term is the first term on the r.h.s. of (20), (21). This means that to the lowest order we neglect the nonlinear terms and the external transverse focusing force.

In this simplified case, and assuming that the electron energy is constant, we can write the equations for the transverse motion as

$$\dot{\beta}_x = -\frac{K_x}{\gamma} (\omega_{rf} - k_{0,1} \beta_z c) \sin(\omega_{rf} t - k_{0,1} z)$$

(23)

$$\dot{\beta}_y = -\frac{K_y}{\gamma} (\omega_{rf} - k_{0,1} \beta_z c) \cos(\omega_{rf} t - k_{0,1} z)$$

(24)

where
\[ K_x = K_0 \cos \left( \frac{\pi \xi}{a} \right), \quad (25) \]
\[ K_y = K_0 \cos \left( \frac{\pi \zeta}{a} \right), \quad (26) \]
\[ K_0 = \frac{e\mu H_0 a c}{\pi mc^2}. \quad (27) \]

To first order we solve for \( \xi, \zeta \) assuming \( K_x = K_y = K_0 \). Later on we will consider the effect of the variation of these quantities with the displacement from axis.

The equation for the energy change is
\[
mc^2 \frac{d\gamma}{dt} = \frac{e\omega_{\gamma} a \mu H_0}{\pi} \{ \beta_x \sin(\omega_{\gamma} t - k_{0,\gamma} z) + \beta_y \cos(\omega_{\gamma} t - k_{0,\gamma} z) \}. \quad (28)
\]

Let us assume that the energy is constant and that
\[ z = \beta_c c t. \quad (29) \]

Then the solution for the transverse motion is
\[ \beta_x = \frac{K_0}{\gamma} \cos(\Delta t), \quad (30) \]
\[ \beta_y = -\frac{K_0}{\gamma} \sin(\Delta t). \quad (31) \]

The oscillation frequency is given by
\[ \Delta = \omega_{\gamma} - k_{0,\gamma} \beta_c, \quad (32) \]

or, using (7) with the choice of the negative sign, by
\[ \Delta = \omega_{\gamma} \sqrt{1 + \beta_c \frac{\pi c}{\omega_{\gamma} a}} \cdot (33) \]

One can easily verify that using (30), (31), (32) equation (28) gives \( \gamma = \text{constant} \).
and that from the same equations it follows that $\beta_z^2 + \beta_y^2 = \frac{K_0^2}{\gamma^2} = \text{constant}$. We then have that also the longitudinal velocity is a constant, and that it can be written as

$$\beta_z = \sqrt{1 - \frac{1 + K_0^2}{\gamma^2}}. \quad (34)$$

This last result justifies using equation (29) to describe the longitudinal motion.

In the same approximation we also have

$$\xi = \frac{K_0}{\gamma} c \sin(\Delta t) \quad (35)$$

$$\zeta = \frac{K_0}{\gamma} c \cos(\Delta t) \quad (36)$$

The trajectory given by the last two equations is a helical trajectory with a period

$$\lambda_U = \frac{2\pi \beta_c c}{\Delta} = \frac{2\pi \beta_c}{\omega_{rf}} \left\{ 1 + \beta \sqrt{1 - \left( \frac{\pi c}{\omega_{rf} a} \right)^2} \right\}. \quad (37)$$

We will call this the (equivalent) undulator period, Notice that this period is shorter that the RF system wavelength, $\lambda_{rf} = 2\pi c / \omega_{rf}$. In the limit $a \to \infty$ and for a relativistic electron one has $\lambda_u \approx \lambda_{rf} / 2$. For a finite value of $a$ the undulator period $\lambda_U$ is between one and one half of the RF wavelength, with the value one obtained at the cutoff frequency, when $a = \pi c / \omega_{rf}$.

To second order, after averaging over the undulator period, the equation of motion become

$$\ddot{\theta}_x = -\frac{K_x}{\gamma} \Delta \sin(\Delta t) + \frac{c}{2} \left( \frac{\pi K_0}{a\gamma} \right)^2 \xi - \frac{\Omega_{uy}^2}{c} \xi, \quad (38)$$

$$\ddot{\pi}_x = -\frac{K_x}{\gamma} \Delta \cos(\Delta t) + \frac{c}{2} \left( \frac{\pi K_0}{a\gamma} \right)^2 \zeta - \frac{\Omega_{uz}^2}{c} \zeta. \quad (39)$$

Notice that the force -due to the RF field- arising from the second r.h.s. terms in (20), (21), is defocusing. In what follows we will assume that we can use the external
focusing force to compensate for the defocusing term and produce the overall amount of focusing that we need for the optimal use of the TWU.

Let us define

\[ \Omega_{b,t}^2 = \Omega_b^2 - \frac{1}{2} \left( \frac{\pi c K_0}{a} \right)^2 \]  

The solution of the equations (38), (39) is given by

\[ \xi = \xi_0 \sin(\Omega_{b,t} t + \varphi_0) + \frac{K_c}{\gamma \Delta} \sin(\Delta t), \]  

\[ \zeta = \zeta_0 \sin(\Omega_{b,t} t + \psi_0) + \frac{K_c}{\gamma \Delta} \cos(\Delta t), \]  

when the condition

\[ \Delta^2 \gg \Omega_b^2 - \frac{1}{2} \left( \frac{\pi c K_0}{a} \right)^2 \]  

is satisfied, as we have assumed.

The last equations show that the trajectory is a superposition of a helical motion with period given by (37), and betatron oscillations with amplitude \( \xi_0, \zeta_0 \), frequency \( \Omega_{bt} \) and a wavelength \( \lambda_b = 2\pi c / \Omega_{b,t} \). Following a common notation in beam optics we will also characterize the betatron motion by the “beta function”, \( \beta_b = \lambda_b / 2\pi = c / \Omega_{b,t} \).

## 5. TWU characteristics

Before discussing the effect of emittance and other parameters on the spectrum and FEL gain we need to select the TWU parameters. We choose to use an RF field with a frequency \( f_r = 12 \text{GHz} \). At or near the frequency we have klystrons with output power in the 50 to 75 MW range, which can be raised to several hundred megawatts by compressing the pulses. A list of the main characteristics for the TWU is given in Table 1.

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<table>
<thead>
<tr>
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<tr>
<td>RF frequency, GH</td>
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</tr>
<tr>
<td>RF power, per mode, MW</td>
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<tr>
<td>Waveguide transverse size, a=b, cm</td>
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<tr>
<td>Equivalent undulator period, ( \lambda_u ), cm</td>
<td>1.45</td>
</tr>
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</table>
Equivalent undulator parameter, \( K_0 \)

<p>| | |</p>
<table>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Attenuation length (copper at ( \sim 300^0 K )), m</td>
<td>36</td>
</tr>
<tr>
<td>Attenuation length (copper at ( \sim 77^0 K )), m</td>
<td>360</td>
</tr>
</tbody>
</table>

Table 1 TWU characteristics

The dependence of the undulator period and the undulator parameter on the waveguide size is shown in Figure 1 and Figure 2. When we reduce the waveguide size toward the cutoff value \( a = \pi c / \omega_r \), the undulator parameter increases, but at the same time the undulator period becomes larger and the attenuation length becomes shorter. In what follows we have made the choice \( a = 1.8 \) cm, and from this we obtain the other values shown in Table 1.

Figure 1 Dependence of the equivalent undulator period on the waveguide size. For the cutoff value \( a = \pi c / \omega_r = 1.25 \) cm, and for relativistic particles, the value of the period is equal to the RF wavelength, 2.5 cm.
The dependence of the undulator parameter on the input radio frequency power is shown in Figure 3.

We can compare the TWU characteristics given in Table 1 with those of a hybrid type static magnetic planar undulator with same period of 1.45 cm. To obtain the same undulator parameter value, $K_{planar} \approx 0.4$, to produce the same wavelength for the same beam energy, this undulator would have a gap of approximately 6.4 mm. This shows that for a small period, and if a large gap is important, the TWU is an interesting alternative to...
the more conventional undulators.


To evaluate the properties of the TWU for the generation of undulator radiation and FELs, and characterize its dependence on the electron beam properties, we will first consider the spectrum on axis, and later the FEL gain.

Consider a TWU being used for the generation of spontaneous undulator radiation. One of its most important characteristics is the spectrum that it generates. In this section we evaluate the spectrum on axis and its dependence on the beam emittance.

The double differential spectrum is given by

\[
\frac{d^2 I}{d\omega d\Omega} = \frac{e^2 \omega^2}{4\pi c} \left| \int dt \, \vec{n} \times (\vec{n} \times \vec{\beta}) \exp[i\omega(t - \frac{\vec{n} \cdot \vec{r}}{c})]\right|^2, \tag{44}
\]

where the integral must be extended over the undulator length. In the absence of betatron oscillation the motion is a helix and the integration of (44) for the spectrum on axis gives the well-known result

\[
\frac{d^2 I}{d\omega d\Omega} = \frac{2r_e m c^2}{c} N_U^2 \gamma^2 \frac{K_0^2}{(1 + K_0^2)^2} \left(\sin \frac{\delta}{\delta}\right)^2, \tag{45}
\]

where \(r_e\) is the classical electron radius, \(N_U\) is the number of periods in the undulator, and

\[
\delta = \pi N_U \frac{\omega - \omega_R}{\omega_R} \tag{46}
\]

The “resonant frequency” is given by the condition that the exponents in the exponential term and in the transverse velocity terms, be equal,

\[
(1 - \beta_z)\omega_R = \Delta \tag{47}
\]

or, using (33),

\[
\omega_R = \frac{\omega_{ph}}{1 - \beta_z} \left(1 + \beta_z \sqrt{1 - (\frac{\pi c}{\omega_{ph} a})^2}\right). \tag{48}
\]

In the relativistic limit we can write \(1 - \beta_z = (1 + K_0^2) / 2\gamma^2\), giving the expression...
\[ \omega_k = \frac{2\gamma^2\omega_{rf}^2}{1 + K_0^2} \left(1 + \beta \sqrt{1 - \left(\frac{\pi c}{\omega_0 a}\right)^2}\right). \]  

(49)

In the limit \( a \to \infty \) the last expression reduces to the well-known Compton backscattering limit \( \omega_R = 4\gamma^2\omega_{rf} / (1 + K_0^2) \).

The main effect of betatron oscillations is to perturb the synchronism condition because of the additional velocity terms, producing a widening of the spectrum with a reduction of the peak value. At the same time they reduce the value of the undulator parameter \( K \). Similarly for the FEL case the effect is to reduce the gain by perturbing the phase of the electron oscillations relative to the electromagnetic field. As we will show in the next sections these effects depend on the choice of the betatron beta function value, and is larger when the beta function is smaller.

In Figure 4 we show the dependence of the spontaneous radiation wavelength on the waveguide transverse size, for the case of 1 GeV beam energy. Other characteristics are given in Table 2, assuming the waveguide parameters of Table 1.

<table>
<thead>
<tr>
<th>Beam energy, ( E/mc^2 )</th>
<th>2000</th>
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<tr>
<td>Spontaneous radiation wavelength, ( \lambda_R ), nm</td>
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</tr>
<tr>
<td>Normalized beam transverse emittance, ( \mu ) m</td>
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</tr>
<tr>
<td>Ratio of emittance to ( \lambda_R / 4\pi )</td>
<td>3</td>
</tr>
</tbody>
</table>

Figure 4 Dependence of the spontaneous radiation wavelength on the waveguide size.
To evaluate the effect of the betatron oscillations on the spectrum we have done a numerical evaluation, as described in Appendix A. The results are shown for two cases in Figure 5 and Figure 6. For the parameter that we have chosen, beam energy and TWU characteristics, the normalized emittance that we assume, 1 mm mrad, gives a value of the ratio of the emittance to the radiation wavelength over $4\pi$ equal to 3. Under this condition we can expect a large effect of the betatron oscillations on the spectrum. This is certainly true when we choose a betatron beta function equal to the “natural” focusing strength, as shown in Figure 5. However if we reduce the focusing the effect on the spectrum is much smaller as shown in Figure 6.

Figure 5 Spectrum on axis for an electron with a betatron amplitude equal to $\sqrt{e\beta_B}$, green line, and 1/10 of that, blue line. The blue line overlaps with the spectrum for the case when the betatron oscillation amplitudes are zero. In this graph we have chosen $\beta_B = \sqrt{2\gamma a / \pi K_0} = 40.9$ m. The ratio of the emittance to the photon space phase density $\lambda_r / 4\pi$ is 1.5.
Figure 6 Spectrum on axis for an electron with a betatron amplitude equal to \( \sqrt{e \beta_B} \), green line, and 1/10 of that, blue line. The blue line overlaps with the spectrum for the case when the betatron oscillation amplitudes are zero. In this graph we have chosen \( \beta_B = 10\sqrt{2} \gamma a / \pi K_0 = 409 \text{ m} \). The ratio of the emittance to the photon space phase density \( \lambda_n / 4\pi \) is 3.

7. TWU based FELs

For short wavelength- high gain FELs, the TWU offers the advantage of a short period and a large aperture. These features, combined with the reduced undulator parameter values, allow an FEL to reach shorter wavelengths for a given beam energy, at a cost of a smaller FEL saturation power. If the TWU waveguide is used also as the electron beam vacuum pipe, its large transverse size will help reduce the effect of the beam-induced wakefields.

In the FEL case the most important parameter is the gain length. As in the case of the spontaneous radiation spectrum, the gain length value tends to increase when the betatron oscillations change the phase of the electrons respect to the electromagnetic wave. However increasing the beta function to reduce the effect of the transverse velocity on the phase change also reduces the electron beam density and thus tends to increase the gain length.

An example of how these effects play to change the gain length is shown for two cases. Case A is a lower energy beam, producing FEL radiation in the water window, from about 2 to 4 nm. We assume for this case a peak current of 2 kA, a normalized transverse emittance of 1 mm mrad, and a relative energy spread of 6x10\(^{-5}\). In case B we use a larger beam energy, to produce FEL radiation in the 0.1 nm spectral region. For case B we increase the peak current to 3.5kA, keeping the same values of the other parameters as in case A. The result shown in Figure 7 to Figure 10 have been calculated using the Xie FEL model.

The results for the water window case are shown in Figure 7, Figure 8. In the first we show the gain length as a function of the betatron focusing for two beam energies,
corresponding to about 2 and 4 nm. The second figure shows the gain length, saturation power and wavelength as a function of energy and for a choice of the betatron focusing $\beta_B = 5m$, corresponding to the minimum shown in Figure 7 for the higher beam energy. One can see that the gain length at 2 nm is about 1.5 m.

Figure 7 Gain length as a function of betatron focusing for the TWU undulator of Table 1, in case A.

Figure 8 Gain length, saturation power and wavelength for case A, the water window FEL. Using the result shown in Figure 7 we have assumed the betatron
focusing to be $\beta_n = 5m$.

In case B, the shorter FEL wavelength, the results are shown in Figure 9 and Figure 10. A wavelength of about 0.1 nm is reached at a beam energy of about 4.8 GeV, with a corresponding gain length of about 6.5 m.

Figure 9 Gain length vs betatron focusing for the TWU undulator and case B, showing a minimum for the 3.8 GeV beam energy near $\beta_n = 50m$.

Figure 10 Gain length, saturation power and wavelength for case B, the X-ray FEL. Using the result shown in Figure 9 we have assumed the betatron focusing to be $\beta_n = 50m$. 
It is important to notice that in all cases considered the saturation power is smaller than what is obtained using a static planar hybrid undulator with a larger undulator parameter and a larger beam energy. The loss in saturation power is due to both factors.

All the results discussed up to now in this section have been evaluated with the Xie’s (7) model. For comparison we show in Figure 11 the result of a Genesis (8) calculation for the case of beam energy 3.8 GeV, and the other parameters as in case B. The results is in general agreement with the previous model.

![Figure 11 Genesis simulation for case B, beam energy 3.8 GeV and radiation wavelength of about 0.15 nm.](image)

8. Effects on power losses in the waveguide walls

Power losses in the waveguide walls reduce the undulator parameter value and hence the radiation wavelength, changing both the spectrum and the gain. If the change is large compared with the spectral line width or the gain width the system performance will be reduced.

The power decreases along the waveguide as

$$P = P_0 \exp(-2\beta_{1,0}z),$$  \hfill (50)

where $\beta_{1,0}$ is given by (18). The power loss will change the radiation wavelength, as shown in Figure 12. The effect can be rather large, even at liquid nitrogen temperature.
Figure 12 Relative change in the radiation wavelength as a function of undulator length. We assume a TWU undulator with the parameters of Table 1. The two curves describe a copper waveguide at room temperature, blue curve, and at 77 °K. In this case the waveguide size is constant.

To estimate the effect of power losses we assume a copper waveguide, with a wall conductivity $\sigma = 6 \times 10^7 \Omega^{-1} m^{-1}$ at room temperature. To reduce the effect we can reduce the wall resistivity by keeping the wall at 77 °K, the liquid nitrogen temperature, or by changing the waveguide size along the undulator length to compensate for the reduce power.

Recent measurements of the conductivity with reduced wall temperature at a frequency of 11.4 Ghz, for a copper waveguide, show that the conductivity increases by about a factor of 3 at about 77 °K, with almost no additional gain by further educing the temperature [9]. In what follows we assume $\sigma = 1.8 \times 10^8 \Omega^{-1} m^{-1}$ at 77 °K.

To discuss the effect of tapering the waveguide transverse size we assume that the change in the waveguide size is given by

$$a = a_0 \exp(-\alpha \beta_{1,0} z)$$  \hspace{1cm} (51)

The parameter $\alpha$ is used to control the amount of change. The effect of changing the waveguide size is shown in Figure 13 and Figure 14 for the case of $\alpha = 0.17$. The relative wavelength change over a 10m long undulator is reduced from about 4% to 1.5%.
Figure 13 Relative change in the waveguide size as a function of undulator length. We assume a TWU undulator with the parameters of Table 1. The two curves describe a copper waveguide at room temperature, blue curve, and at 77 °K. In this case the waveguide size changes according to (51) with $\alpha=0.17$.

Figure 14 Relative change in the radiation wavelength as a function of undulator length. We assume a TWU undulator with the parameters of Table 1. The two curves describe a copper waveguide at room temperature, blue curve, and at 77 °K. In this case the waveguide changes according to (51) with $\alpha=0.17$. We have assumed a beam energy of 1.05 GeV, corresponding to a central radiation wavelength of 2 nm.

If we increase the value of $\alpha$ to 0.28 we see that the wavelength change changes
sign, instead of decreasing the wavelength increases along the undulator, as shown in Figure 15 and 16. The relative change is also reduced to 0.14%. These results show that changing the waveguide size along the undulator gives a very useful way to control the wavelength change and reduce it to acceptable values.

**Figure 15** Relative change in the waveguide size as a function of undulator length. We assume a TWU undulator with the parameters of Table 1. The two curves describe a copper waveguide at room temperature, blue curve, and at 77 °K. In this case the waveguide size changes according to (51) with $\alpha = 0.28$.

**Figure 16** Relative change in the radiation wavelength as a function of undulator length. We assume a TWU undulator with the parameters of Table 1. The
two curves describe a copper waveguide at room temperature, blue curve, and at 77 °K. In this case the waveguide changes according to (51) with \( \alpha = 0.28 \). We have assumed a beam energy of 1.05 GeV, corresponding to a central radiation wavelength of 2 nm.

9. Conclusions

We have discussed the characteristics of the TWU and its possible usefulness for the generation of spontaneous undulator radiation and for short wavelength FELs. The recent development of high power X-band klystrons, and the possibility of using compression to reach power levels in hundred Mw range, make the TWU a realistic system, offering the advantage of short period and large aperture. The disadvantage is a lower radiation power level for spontaneous radiation and FELs, due to the small value of the undulator parameter and the lower beam energy for a given wavelength, when compared to the more conventional approach.

However, when the beam energy is limited, or when one would like to push the radiation wavelength to lower limits, the TWU can be considered as a possible alternative to static magnetic undulator. It is also interesting to notice that by changing the ratio between the amplitudes of the two modes in the waveguide one change the trajectory from helical to planar, going through the intermediate helical with elliptical projection on the transverse plane. This offers an easy way to change the polarization of the emitted wave from circular to elliptical to linear.

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\begin{itemize}
\item[T5] S.G. Tantawi et al., “A High-power Multimode X-Band RF Pulse Compression System for Future Linear Colliders”,
\end{itemize}
9 S. Tantawi, private communication.