# Position Dependence Of The $K$ Parameter In The Delta Undulator

Zachary Wolf
SLAC

January 19, 2016

**Abstract**

In order to understand the alignment tolerances of the Delta undulator, we must know how the $K$ value depends on the transverse position of the beam. In this note we calculate the $K$ value as a function of beam position for different polarization modes and row shift settings. The effect of transverse field rolloff of the quadrants is included, and we will see that it plays a large role in the position dependence of $K$. In particular, it gives a gradient to $K$ at the magnetic center in the elliptical modes, potentially making the undulator alignment requirements difficult to meet.

## 1 Introduction[1]

The position dependence of the $K$ parameter in the Delta undulator determines how accurately the undulator must be aligned in order to be resonant with other undulators. In an EPU the $K$ parameter depends on position differently for different quadrant row settings. In addition, the $K$ parameter depends on the shape of the quadrant fields. In this note we explore these dependencies.

The organization of the note is as follows. We first calculate the magnetic fields in the Delta undulator. This is done in a rotated coordinate frame relative to the laboratory frame in which the undulator is used. Rotating the coordinates does not affect the calculated $K$ value and it simplifies the calculations. From the fields, we calculate the trajectory slopes which we use to calculate the slippage. The slope of the linear part of the slippage change with longitudinal position lets us calculate the $K$ value of the undulator. The $K$ value is calculated at different transverse positions and at different undulator row settings in order to study the position dependence of $K$.

## 2 Magnetic Field In The Undulator

### 2.1 The Scalar Potential

The scalar potential from a single magnet array satisfies Laplace's equation. It varies sinusoidally down the array, it decreases away from the array, and it falls off as one moves away from the center line of the array in the transverse direction. In the coordinate system shown in figure 1, the scalar potential can be expressed as a Fourier sum of terms with a fundamental term of the form

$$\phi = \phi_0 \cos(k_s s) \exp\left(-k_r r\right) \cos\left(k_u \left(z - z_0\right)\right) \tag{1}$$

[1] Work supported in part by the DOE Contract DE-AC02-76SF00515. This work was performed in support of the LCLS project at SLAC.

where $\phi_0$ is a constant, $k_u = 2\pi/\lambda_u$ where $\lambda_u$ is the undulator period, $z$ is the coordinate down the undulator, and $z_0$ gives the quadrant position along $z$. $k_r$ and $k_s$ determine the behavior of the potential in the transverse directions. The potential decreases as one moves radially away from the magnet and this is expressed by $\exp(-k_r r)$. The potential also decreases as one moves to the side and this is given by the $\cos(k_s s)$ dependence over a limited range of $s$. Since the Laplacian of the potential is zero, we have the constraint

$$-k_s^2 + k_r^2 - k_u^2 = 0 \tag{2}$$

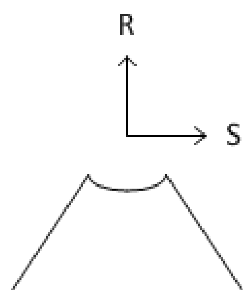


Figure 1: Coordinate system for the scalar potential of a single quadrant.

Although the scalar potential is a sum of terms, we assume that the fundamental term dominates the $K$ value. In a Fourier expansion of this type, typically the amplitude of the fundamental term is the largest. Furthermore, harmonics with a $\cos(nk_u(z - z_0))$ dependence give contributions to the transverse velocity of the beam that go as $1/n$, and therefore the contributions to $K$ from these harmonics go as $1/n$. The smaller amplitudes of the higher harmonics combined with the $1/n$ dependence lead us to expect their contribution to $K$ will be small. The scalar potential also has terms with different $k_s$ values (and different $k_r$ values as given by the constraint). Satisfactory fits of $B_r = \partial\phi/\partial r$ as a function of $s$ above a magnet array can be done with only a few $\cos(k_s s)$ terms. The fundamental term is the largest and has a value of $k_s$ determined by the effective width of the magnet block. Additional terms have $k_s$ values which are not integer multiples of the fundamental, but are determined by the detailed magnet shape. These higher order terms are typically much smaller than the fundamental. The higher order terms may be useful to construct a precise field behavior by shaping the magnet. But for now, we consider only the fundamental term which dominates $K$.

We use the form of the potential given in equation 1 to calculate the magnetic scalar potential in the undulator by rotating the four quadrants and summing their rotated scalar potentials. The Delta undulator is oriented as shown on the left side of figure 2. In the laboratory, $z$ is along the beam direction, $y_L$ is up, and $x_L$ makes a right handed system. For our calculations, it is more convenient to use the rotated system $x$, $y$, $z$, where $x$ is in the direction from quadrant 3 to quadrant 1, $y$ is in the direction from quadrant 4 to quadrant 2, and $z$ is in the beam direction. Using equation 1, the scalar potential for each of the quadrants in the $x$, $y$, $z$ system is

$$\phi_1(x, y, z) = \phi_{0Q} \cos(k_s y) \exp(k_r x) \cos(k_u (z - z_{01})) \tag{3}$$

$$\phi_2(x, y, z) = \phi_{0Q} \cos(k_s x) \exp(k_r y) \cos(k_u (z - z_{02})) \tag{4}$$

$$\phi_3(x, y, z) = -\phi_{0Q} \cos(k_s y) \exp(-k_r x) \cos(k_u (z - z_{03})) \tag{5}$$

$$\phi_4(x, y, z) = -\phi_{0Q} \cos(k_s x) \exp(-k_r y) \cos(k_u (z - z_{04})) \tag{6}$$

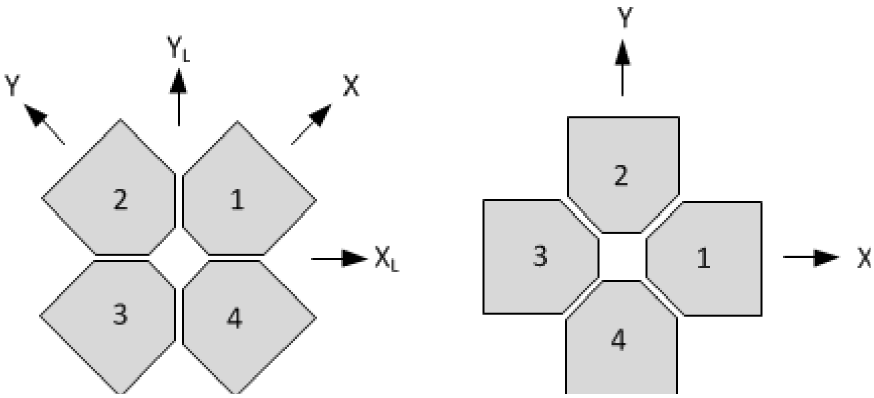


Figure 2: The left side of the figure shows the Delta undulator in its configuration in the tunnel where $y_L$ is up, $z$ is in the beam direction, and $x_L$ makes a right handed system. For our calculations, we use the rotated coordinate system on the right, where $x$ is along the line pointing from quadrant 3 to quadrant 1, $y$ is along the line pointing from quadrant 4 to quadrant 2, and $z$ makes a right handed system.

where $z_{0i}$ is the longitudinal shift of quadrant $i$, and $\phi_{0Q}$ is the amplitude of the scalar potential of all the identical quadrants on the axis of the undulator where $x = 0$ and $y = 0$. Quadrants 3 and 4 are loaded with opposite polarity magnets as quadrants 1 and 2 in order to make a vertical field planar undulator in the laboratory frame when all the rows are aligned. This accounts for the minus signs, $-\phi_{0Q}$, in the potentials for quadrants 3 and 4.

The scalar potential for the undulator is the sum of the scalar potentials for the quadrants. Quadrants 1 and 3 both depend on $x$, and quadrants 2 and 4 both depend on $y$. We first add the scalar potentials for quadrants 1 and 3, and then add the scalar potentials for quadrants 2 and 4, and then add the sums to get the scalar potential for the whole undulator. We will interpret this as forming the entire undulator from two crossed planar adjustable phase undulators.

The scalar potential for the combination of quadrants 1 and 3 is given by

$$\phi_{13} = \phi_{0Q} \cos(k_s y) \exp(k_r x) \cos(k_u (z - z_{01})) - \phi_{0Q} \cos(k_s y) \exp(-k_r x) \cos(k_u (z - z_{03})) \tag{7}$$

Let

$$z_{01} = Z_{13} + \frac{\Delta_{13}}{2} \tag{8}$$

$$z_{03} = Z_{13} - \frac{\Delta_{13}}{2} \tag{9}$$

where

$$Z_{13} = \frac{z_{01} + z_{03}}{2} \tag{10}$$

is the average z-position of the quadrants, and

$$\Delta_{13} = z_{01} - z_{03} \tag{11}$$

is the z-shift between the quadrants. With these definitions, the scalar potential for the pair of quadrants becomes

$$\begin{aligned} \phi_{13} = {} & 2\phi_{0Q} \cos(k_s y) \sinh(k_r x) \cos\left(k_u \frac{\Delta_{13}}{2}\right) \cos(k_u (z - Z_{13})) \\ & +2\phi_{0Q} \cos(k_s y) \cosh(k_r x) \sin\left(k_u \frac{\Delta_{13}}{2}\right) \sin(k_u (z - Z_{13})) \end{aligned} \tag{12}$$

This is the scalar potential for a planar adjustable phase undulator[2]. The full range of amplitudes and phases are covered if the range of $\Delta_{13}$ includes $\left[-\frac{\lambda_u}{2}, \frac{\lambda_u}{2}\right]$ and the range of $Z_{13}$ includes $\left[-\frac{\lambda_u}{2}, \frac{\lambda_u}{2}\right]$.

Similarly, the scalar potential for the combination of quadrants 2 and 4 is given by

$$\phi_{24} = \phi_{0Q} \cos(k_s x) \exp\left(k_r y\right) \cos\left(k_u \left(z - z_{02}\right)\right) - \phi_{0Q} \cos(k_s x) \exp\left(-k_r y\right) \cos\left(k_u \left(z - z_{04}\right)\right) \tag{13}$$

Let

$$z_{02} = Z_{24} + \frac{\Delta_{24}}{2} \tag{14}$$

$$z_{04} = Z_{24} - \frac{\Delta_{24}}{2} \tag{15}$$

where

$$Z_{24} = \frac{z_{02} + z_{04}}{2} \tag{16}$$

is the average z-position of the quadrants, and

$$\Delta_{24} = z_{02} - z_{04} \tag{17}$$

is the z-shift between the quadrants.

With these definitions, the scalar potential for the pair of quadrants becomes

$$\begin{aligned} \phi_{24} = \; & 2\phi_{0Q} \cos(k_s x) \sinh\left(k_r y\right) \cos\left(k_u \frac{\Delta_{24}}{2}\right) \cos\left(k_u \left(z - Z_{24}\right)\right) \\ & + 2\phi_{0Q} \cos(k_s x) \cosh\left(k_r y\right) \sin\left(k_u \frac{\Delta_{24}}{2}\right) \sin\left(k_u \left(z - Z_{24}\right)\right) \end{aligned} \tag{18}$$

This is again the potential for a planar adjustable phase undulator.

The scalar potential for the undulator is the sum of the scalar potentials for the quadrant pairs.

$$\phi = \phi_{13} + \phi_{24} \tag{19}$$

Performing the sum, we find

$$\begin{aligned} \phi = \; & 2\phi_{0Q} \cos(k_s y) \sinh\left(k_r x\right) \cos\left(k_u \frac{\Delta_{13}}{2}\right) \cos\left(k_u \left(z - Z_{13}\right)\right) \\ & + 2\phi_{0Q} \cos(k_s y) \cosh\left(k_r x\right) \sin\left(k_u \frac{\Delta_{13}}{2}\right) \sin\left(k_u \left(z - Z_{13}\right)\right) \\ & + 2\phi_{0Q} \cos(k_s x) \sinh\left(k_r y\right) \cos\left(k_u \frac{\Delta_{24}}{2}\right) \cos\left(k_u \left(z - Z_{24}\right)\right) \\ & + 2\phi_{0Q} \cos(k_s x) \cosh\left(k_r y\right) \sin\left(k_u \frac{\Delta_{24}}{2}\right) \sin\left(k_u \left(z - Z_{24}\right)\right) \end{aligned} \tag{20}$$

By putting in the various values for $Z_{13}$, $Z_{24}$, $\Delta_{13}$, and $\Delta_{24}$, we get the scalar potential for the various undulator modes at different $K$ values.

[2] Z. Wolf, "Variable Phase PPM Undulator Study", LCLS-TN-11-1, May, 2011.

## 2.2 Magnetic Field In The Undulator Coordinate System

The magnetic field in the undulator is given by $B = \nabla\phi$. Taking the gradient we find

$$\begin{aligned} B_x(x,y,z) &= 2\phi_{0Q}k_u[\frac{k_r}{k_u}\cos\left(k_u\frac{\Delta_{13}}{2}\right)\cos(k_s y)\cosh(k_r x)\cos(k_u(z-Z_{13})) \\ &+\frac{k_r}{k_u}\sin\left(k_u\frac{\Delta_{13}}{2}\right)\cos(k_s y)\sinh(k_r x)\sin(k_u(z-Z_{13})) \\ &-\frac{k_s}{k_u}\cos\left(k_u\frac{\Delta_{24}}{2}\right)\sin(k_s x)\sinh(k_r y)\cos(k_u(z-Z_{24})) \\ &-\frac{k_s}{k_u}\sin\left(k_u\frac{\Delta_{24}}{2}\right)\sin(k_s x)\cosh(k_r y)\sin(k_u(z-Z_{24}))] \end{aligned} \tag{21}$$

$$\begin{aligned} B_y(x,y,z) &= 2\phi_{0Q}k_u[-\frac{k_s}{k_u}\cos\left(k_u\frac{\Delta_{13}}{2}\right)\sin(k_s y)\sinh(k_r x)\cos(k_u(z-Z_{13})) \\ &-\frac{k_s}{k_u}\sin\left(k_u\frac{\Delta_{13}}{2}\right)\sin(k_s y)\cosh(k_r x)\sin(k_u(z-Z_{13})) \\ &+\frac{k_r}{k_u}\cos\left(k_u\frac{\Delta_{24}}{2}\right)\cos(k_s x)\cosh(k_r y)\cos(k_u(z-Z_{24})) \\ &+\frac{k_r}{k_u}\sin\left(k_u\frac{\Delta_{24}}{2}\right)\cos(k_s x)\sinh(k_r y)\sin(k_u(z-Z_{24}))] \end{aligned} \tag{22}$$

$$\begin{aligned} B_z(x,y,z) &= 2\phi_{0Q}k_u[-\cos\left(k_u\frac{\Delta_{13}}{2}\right)\cos(k_s y)\sinh(k_r x)\sin(k_u(z-Z_{13})) \\ &+\sin\left(k_u\frac{\Delta_{13}}{2}\right)\cos(k_s y)\cosh(k_r x)\cos(k_u(z-Z_{13})) \\ &-\cos\left(k_u\frac{\Delta_{24}}{2}\right)\cos(k_s x)\sinh(k_r y)\sin(k_u(z-Z_{24})) \\ &+\sin\left(k_u\frac{\Delta_{24}}{2}\right)\cos(k_s x)\cosh(k_r y)\cos(k_u(z-Z_{24}))] \end{aligned} \tag{23}$$

In order to simplify these formulas further, let

$$B_0 = 2\phi_{0Q}k_u \tag{24}$$

$$B_{xc} = B_0\cos\left(k_u\frac{\Delta_{13}}{2}\right) \tag{25}$$

$$B_{xs} = B_0\sin\left(k_u\frac{\Delta_{13}}{2}\right) \tag{26}$$

$$B_{yc} = B_0\cos\left(k_u\frac{\Delta_{24}}{2}\right) \tag{27}$$

$$B_{ys} = B_0\sin\left(k_u\frac{\Delta_{24}}{2}\right) \tag{28}$$

With these substitutions, the fields become

$$
\begin{aligned}
B_x(x,y,z) &= \frac{k_r}{k_u} B_{xc} \cos(k_s y) \cosh(k_r x) \cos(k_u (z - Z_{13})) \\
&+ \frac{k_r}{k_u} B_{xs} \cos(k_s y) \sinh(k_r x) \sin(k_u (z - Z_{13})) \\
&- \frac{k_s}{k_u} B_{yc} \sin(k_s x) \sinh(k_r y) \cos(k_u (z - Z_{24})) \\
&- \frac{k_s}{k_u} B_{ys} \sin(k_s x) \cosh(k_r y) \sin(k_u (z - Z_{24}))
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
B_y(x,y,z) &= -\frac{k_s}{k_u} B_{xc} \sin(k_s y) \sinh(k_r x) \cos(k_u (z - Z_{13})) \\
&- \frac{k_s}{k_u} B_{xs} \sin(k_s y) \cosh(k_r x) \sin(k_u (z - Z_{13})) \\
&+ \frac{k_r}{k_u} B_{yc} \cos(k_s x) \cosh(k_r y) \cos(k_u (z - Z_{24})) \\
&+ \frac{k_r}{k_u} B_{ys} \cos(k_s x) \sinh(k_r y) \sin(k_u (z - Z_{24}))
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
B_z(x,y,z) &= -B_{xc} \cos(k_s y) \sinh(k_r x) \sin(k_u (z - Z_{13})) \\
&+ B_{xs} \cos(k_s y) \cosh(k_r x) \cos(k_u (z - Z_{13})) \\
&- B_{yc} \cos(k_s x) \sinh(k_r y) \sin(k_u (z - Z_{24})) \\
&+ B_{ys} \cos(k_s x) \cosh(k_r y) \cos(k_u (z - Z_{24}))
\end{aligned} \tag{31}
$$

We can change the origin of $z$ such that

$$z = z' + Z_{13} \tag{32}$$

If we let

$$\delta = k_u (Z_{13} - Z_{24}) \tag{33}$$

and drop the prime, we write the fields as a function of position as

$$
\begin{aligned}
B_x(x,y,z) &= B_{xc} \frac{k_r}{k_u} \cos(k_s y) \cosh(k_r x) \cos(k_u z) \\
&+ B_{xs} \frac{k_r}{k_u} \cos(k_s y) \sinh(k_r x) \sin(k_u z) \\
&- B_{yc} \frac{k_s}{k_u} \sin(k_s x) \sinh(k_r y) \cos(k_u z + \delta) \\
&- B_{ys} \frac{k_s}{k_u} \sin(k_s x) \cosh(k_r y) \sin(k_u z + \delta)
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
B_y(x,y,z) &= -B_{xc} \frac{k_s}{k_u} \sin(k_s y) \sinh(k_r x) \cos(k_u z) \\
&- B_{xs} \frac{k_s}{k_u} \sin(k_s y) \cosh(k_r x) \sin(k_u z) \\
&+ B_{yc} \frac{k_r}{k_u} \cos(k_s x) \cosh(k_r y) \cos(k_u z + \delta) \\
&+ B_{ys} \frac{k_r}{k_u} \cos(k_s x) \sinh(k_r y) \sin(k_u z + \delta)
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
B_z(x,y,z) &= -B_{xc}\cos(k_s y)\sinh(k_r x)\sin(k_u z) \\
&\quad +B_{xs}\cos(k_s y)\cosh(k_r x)\cos(k_u z) \\
&\quad -B_{yc}\cos(k_s x)\sinh(k_r y)\sin(k_u z+\delta) \\
&\quad +B_{ys}\cos(k_s x)\cosh(k_r y)\cos(k_u z+\delta) \qquad (36)
\end{aligned}
$$

# 3 Trajectories In The Undulator

## 3.1 Equations Of Motion

The trajectory of a charged particle beam in the undulator is determined by the Lorentz force law

$$
\frac{d\mathbf{p}}{dt} = q(\mathbf{v}\times\mathbf{B}) \qquad (37)
$$

where $\mathbf{p} = \gamma m\mathbf{v}$, $\gamma = (1-\beta^2)^{-1}$, and $m$ is the particle rest mass. The energy of the particle is constant in the magnetic field of the undulator, except for radiation losses which we neglect. With constant energy, $\gamma$ is constant. The Lorentz force law becomes

$$
\dot{\mathbf{v}} = \frac{q}{\gamma m}(\mathbf{v}\times\mathbf{B}) \qquad (38)
$$

With the substitution $d/dt = v_z d/dz$, the changes in the individual velocity components with $z$ are given by

$$
\begin{aligned}
\frac{dv_x}{dz} &= \frac{q}{\gamma m v_z}(v_y B_z - v_z B_y) \qquad (39)\\
\frac{dv_y}{dz} &= \frac{q}{\gamma m v_z}(v_z B_x - v_x B_z) \qquad (40)\\
\frac{dv_z}{dz} &= \frac{q}{\gamma m v_z}(v_x B_y - v_y B_x) \qquad (41)
\end{aligned}
$$

## 3.2 Iterative Solution

We solve these equations iteratively by expanding in powers of a small parameter. Let

$$
\epsilon = \frac{qB_0}{\gamma m k_u c} \qquad (42)
$$

be the small dimensionless expansion parameter. In terms of this parameter, the equations of motion are

$$
\begin{aligned}
\frac{dv_x}{dz} &= \epsilon\frac{k_u c}{B_0 v_z}(v_y B_z - v_z B_y) \qquad (43)\\
\frac{dv_y}{dz} &= \epsilon\frac{k_u c}{B_0 v_z}(v_z B_x - v_x B_z) \qquad (44)\\
\frac{dv_z}{dz} &= \epsilon\frac{k_u c}{B_0 v_z}(v_x B_y - v_y B_x) \qquad (45)
\end{aligned}
$$

At this point we could divide all velocities by $c$, divide all positions by $1/k_u$, and divide all magnetic fields by $B_0$ in order to eliminate the factor $\frac{k_u c}{B_0 v_z}$. We think of small quantities in terms of these dimensionless variables. In the interest of clarity, however, we keep the factor and continue to use mks units, so there is no confusion about the meaning of a quantity.

Expanding the particle position in terms of the expansion parameter, we have

$$x = x_{(0)} + \epsilon x_{(1)} + \epsilon^2 x_{(2)} + \cdots \tag{46}$$
$$y = y_{(0)} + \epsilon y_{(1)} + \epsilon^2 y_{(2)} + \cdots \tag{47}$$
$$z = z \tag{48}$$

where the last line indicates that the z-position of the particle is the independent variable. The numbers in parenthesis indicate the order of the expansion. The velocities are

$$v_x = v_{x(0)} + \epsilon v_{x(1)} + \epsilon^2 v_{x(2)} + \cdots \tag{49}$$
$$v_y = v_{y(0)} + \epsilon v_{y(1)} + \epsilon^2 v_{y(2)} + \cdots \tag{50}$$
$$v_z = v_{z(0)} + \epsilon v_{z(1)} + \epsilon^2 v_{z(2)} + \cdots \tag{51}$$

The magnetic fields are given by

$$B_x = B_x|_0 + \epsilon \partial_x B_x|_0 x_{(1)} + \epsilon \partial_y B_x|_0 y_{(1)} + \cdots \tag{52}$$
$$B_y = B_y|_0 + \epsilon \partial_x B_y|_0 x_{(1)} + \epsilon \partial_y B_y|_0 y_{(1)} + \cdots \tag{53}$$
$$B_z = B_z|_0 + \epsilon \partial_x B_z|_0 x_{(1)} + \epsilon \partial_y B_z|_0 y_{(1)} + \cdots \tag{54}$$

where

$$B_i|_0 = B_i(x_{(0)}, y_{(0)}, z) \tag{55}$$
$$\partial_j B_i|_0 = \frac{\partial B_i}{\partial x_j}(x_{(0)}, y_{(0)}, z) \tag{56}$$

where $i = x, y, z$, $j = x, y$.

The transverse positions are found from the velocities as follows:

$$v_x = \frac{dx}{dz}\frac{dz}{dt} \tag{57}$$

$$\begin{aligned} x(z) &= x_0 + \int_0^z v_x(z') \frac{1}{v_z(z')} dz' \\ &= x_0 + \int_0^z \left(v_{x(0)} + \epsilon v_{x(1)} + \cdots\right) \frac{1}{v_{z(0)}} \left(1 - \epsilon \frac{v_{z(1)}}{v_{z(0)}} + \cdots\right) dz' \\ &= x_0 + \frac{v_{x(0)}}{v_{z(0)}} z + \epsilon \int_0^z \left(\frac{v_{x(1)}}{v_{z(0)}} - \frac{v_{x(0)} v_{z(1)}}{v_{z(0)}^2}\right) dz' + \cdots \end{aligned} \tag{58}$$

In these equations $x_0$ is the constant initial position where the beam enters the undulator. Similarly

$$y(z) = y_0 + \frac{v_{y(0)}}{v_{z(0)}} z + \epsilon \int_0^z \left(\frac{v_{y(1)}}{v_{z(0)}} - \frac{v_{y(0)} v_{z(1)}}{v_{z(0)}^2}\right) dz' + \cdots \tag{59}$$

We see that

$$x_{(0)} = x_0 + \frac{v_{x(0)}}{v_{z(0)}} z \tag{60}$$
$$y_{(0)} = y_0 + \frac{v_{y(0)}}{v_{z(0)}} z \tag{61}$$

and

$$x_{(1)} = \int_0^z \left(\frac{v_{x(1)}}{v_{z(0)}} - \frac{v_{x(0)} v_{z(1)}}{v_{z(0)}^2}\right) dz' \tag{62}$$
$$y_{(1)} = \int_0^z \left(\frac{v_{y(1)}}{v_{z(0)}} - \frac{v_{y(0)} v_{z(1)}}{v_{z(0)}^2}\right) dz' \tag{63}$$

With the expansions in the small parameter $\epsilon$, the equations of motion become

$$\begin{aligned}
& \frac{dv_{x(0)}}{dz} + \epsilon \frac{dv_{x(1)}}{dz} + \epsilon^2 \frac{dv_{x(2)}}{dz} + \cdots \\
= \; & \epsilon \frac{k_u c}{B_0 v_{z(0)}} \left[ 1 - \epsilon \frac{v_{z(1)}}{v_{z(0)}} + \cdots \right] \\
& \times \{ (v_{y(0)} + \epsilon v_{y(1)} + \cdots)(B_z|_0 + \epsilon \partial_x B_z|_0 x_{(1)} + \epsilon \partial_y B_z|_0 y_{(1)} + \cdots) \\
& - (v_{z(0)} + \epsilon v_{z(1)} + \cdots)(B_y|_0 + \epsilon \partial_x B_y|_0 x_{(1)} + \epsilon \partial_y B_y|_0 y_{(1)} + \cdots) \}
\end{aligned} \tag{64}$$

$$\begin{aligned}
& \frac{dv_{y(0)}}{dz} + \epsilon \frac{dv_{y(1)}}{dz} + \epsilon^2 \frac{dv_{y(2)}}{dz} + \cdots \\
= \; & \epsilon \frac{k_u c}{B_0 v_{z(0)}} \left[ 1 - \epsilon \frac{v_{z(1)}}{v_{z(0)}} + \cdots \right] \\
& \times \{ (v_{z(0)} + \epsilon v_{z(1)} + \cdots)(B_x|_0 + \epsilon \partial_x B_x|_0 x_{(1)} + \epsilon \partial_y B_x|_0 y_{(1)} + \cdots) \\
& - (v_{x(0)} + \epsilon v_{x(1)} + \cdots)(B_z|_0 + \epsilon \partial_x B_z|_0 x_{(1)} + \epsilon \partial_y B_z|_0 y_{(1)} + \cdots) \}
\end{aligned} \tag{65}$$

$$\begin{aligned}
& \frac{dv_{z(0)}}{dz} + \epsilon \frac{dv_{z(1)}}{dz} + \epsilon^2 \frac{dv_{z(2)}}{dz} + \cdots \\
= \; & \epsilon \frac{k_u c}{B_0 v_{z(0)}} \left[ 1 - \epsilon \frac{v_{z(1)}}{v_{z(0)}} + \cdots \right] \\
& \times \{ (v_{x(0)} + \epsilon v_{x(1)} + \cdots)(B_y|_0 + \epsilon \partial_x B_y|_0 x_{(1)} + \epsilon \partial_y B_y|_0 y_{(1)} + \cdots) \\
& - (v_{y(0)} + \epsilon v_{y(1)} + \cdots)(B_x|_0 + \epsilon \partial_x B_x|_0 x_{(1)} + \epsilon \partial_y B_x|_0 y_{(1)} + \cdots) \}
\end{aligned} \tag{66}$$

We now proceed to solve these equations for zeroth, and first orders.

#### 3.2.1 Zeroth Order

To zeroth order in $\epsilon$, the equations of motion are

$$\frac{dv_{x(0)}}{dz} = 0 \tag{67}$$

$$\frac{dv_{y(0)}}{dz} = 0 \tag{68}$$

$$\frac{dv_{z(0)}}{dz} = 0 \tag{69}$$

The solutions can be written by inspection

$$v_{x(0)} = v_{x0} \tag{70}$$

$$v_{y(0)} = v_{y0} \tag{71}$$

$$v_{z(0)} = v_{z0} \tag{72}$$

where the $v_{x0}$, $v_{y0}$, $v_{z0}$ are the constant initial velocities. The corresponding zeroth order transverse positions are

$$x_{(0)} = x_0 + \frac{v_{x0}}{v_{z0}} z \tag{73}$$

$$y_{(0)} = y_0 + \frac{v_{y0}}{v_{z0}} z \tag{74}$$

where the $x_0$ and $y_0$ have been previously defined to be the initial positions.

At this point, we make a simplifying assumption. We assume that all particles have zero initial transverse velocity. In this case

$$\begin{aligned} v_{x(0)} &= 0 && (75)\\ v_{y(0)} &= 0 && (76)\\ v_{z(0)} &= v_{z0} && (77) \end{aligned}$$

and

$$\begin{aligned} x_{(0)} &= x_0 && (78)\\ y_{(0)} &= y_0 && (79) \end{aligned}$$

### 3.2.2 First Order

To first order in $\epsilon$, the equations of motion are

$$\begin{aligned} \frac{dv_{x(1)}}{dz} &= \frac{k_u c}{B_0 v_{z(0)}}\{v_{y(0)} B_z|_0 - v_{z(0)} B_y|_0\} && (80)\\ \frac{dv_{y(1)}}{dz} &= \frac{k_u c}{B_0 v_{z(0)}}\{v_{z(0)} B_x|_0 - v_{x(0)} B_z|_0\} && (81)\\ \frac{dv_{z(1)}}{dz} &= \frac{k_u c}{B_0 v_{z(0)}}\{v_{x(0)} B_y|_0 - v_{y(0)} B_x|_0\} && (82) \end{aligned}$$

With zero initial transverse velocity, these equations become

$$\begin{aligned} \frac{dv_{x(1)}}{dz} &= -\frac{k_u c}{B_0} B_y|_0 && (83)\\ \frac{dv_{y(1)}}{dz} &= \frac{k_u c}{B_0} B_x|_0 && (84)\\ \frac{dv_{z(1)}}{dz} &= 0 && (85) \end{aligned}$$

Inserting the expressions for the fields in the undulator coordinate system, we have

$$\begin{aligned} \frac{dv_{x(1)}}{dz} = -\frac{k_u c}{B_0}[&-B_{xc}\frac{k_s}{k_u}\sin(k_s y_0)\sinh(k_r x_0)\cos(k_u z)\\ &-B_{xs}\frac{k_s}{k_u}\sin(k_s y_0)\cosh(k_r x_0)\sin(k_u z)\\ &+B_{yc}\frac{k_r}{k_u}\cos(k_s x_0)\cosh(k_r y_0)\cos(k_u z+\delta)\\ &+B_{ys}\frac{k_r}{k_u}\cos(k_s x_0)\sinh(k_r y_0)\sin(k_u z+\delta)] \qquad (86) \end{aligned}$$

$$\begin{aligned} \frac{dv_{y(1)}}{dz} = \frac{k_u c}{B_0}[&B_{xc}\frac{k_r}{k_u}\cos(k_s y_0)\cosh(k_r x_0)\cos(k_u z)\\ &+B_{xs}\frac{k_r}{k_u}\cos(k_s y_0)\sinh(k_r x_0)\sin(k_u z)\\ &-B_{yc}\frac{k_s}{k_u}\sin(k_s x_0)\sinh(k_r y_0)\cos(k_u z+\delta)\\ &-B_{ys}\frac{k_s}{k_u}\sin(k_s x_0)\cosh(k_r y_0)\sin(k_u z+\delta) \qquad (87) \end{aligned}$$

$$\frac{dv_{z(1)}}{dz} = 0 \tag{88}$$

We integrate to find the velocity components to first order:

$$\begin{aligned} v_{x(1)} &= -\frac{c}{B_0}[-B_{xc}\frac{k_s}{k_u}\sin(k_s y_0)\sinh(k_r x_0)\sin(k_u z) \\ &+B_{xs}\frac{k_s}{k_u}\sin(k_s y_0)\cosh(k_r x_0)\cos(k_u z) \\ &+B_{yc}\frac{k_r}{k_u}\cos(k_s x_0)\cosh(k_r y_0)\sin(k_u z+\delta) \\ &-B_{ys}\frac{k_r}{k_u}\cos(k_s x_0)\sinh(k_r y_0)\cos(k_u z+\delta)] \end{aligned} \tag{89}$$

$$\begin{aligned} v_{y(1)} &= \frac{c}{B_0}[B_{xc}\frac{k_r}{k_u}\cos(k_s y_0)\cosh(k_r x_0)\sin(k_u z) \\ &-B_{xs}\frac{k_r}{k_u}\cos(k_s y_0)\sinh(k_r x_0)\cos(k_u z) \\ &-B_{yc}\frac{k_s}{k_u}\sin(k_s x_0)\sinh(k_r y_0)\sin(k_u z+\delta) \\ &+B_{ys}\frac{k_s}{k_u}\sin(k_s x_0)\cosh(k_r y_0)\cos(k_u z+\delta) \end{aligned} \tag{90}$$

$$v_{z(1)} = 0 \tag{91}$$

## 3.3 Solution To The Equations Of Motion

The velocity of a particle in the undulator can now be given to first order. Using

$$v_x = v_{x(0)} + \epsilon v_{x(1)} + \cdots \tag{92}$$
$$v_y = v_{y(0)} + \epsilon v_{y(1)} + \cdots \tag{93}$$
$$v_z = v_{z(0)} + \epsilon v_{z(1)} + \cdots \tag{94}$$

with

$$\epsilon = \frac{qB_0}{\gamma m k_u c} \tag{95}$$

we get

$$\begin{aligned} v_x &= -\frac{q}{\gamma m k_u}[-B_{xc}\frac{k_s}{k_u}\sin(k_s y_0)\sinh(k_r x_0)\sin(k_u z) \\ &+B_{xs}\frac{k_s}{k_u}\sin(k_s y_0)\cosh(k_r x_0)\cos(k_u z) \\ &+B_{yc}\frac{k_r}{k_u}\cos(k_s x_0)\cosh(k_r y_0)\sin(k_u z+\delta) \\ &-B_{ys}\frac{k_r}{k_u}\cos(k_s x_0)\sinh(k_r y_0)\cos(k_u z+\delta)] \end{aligned} \tag{96}$$

$$
\begin{aligned}
v_y &= \frac{q}{\gamma m k_u}[B_{xc}\frac{k_r}{k_u}\cos(k_s y_0)\cosh(k_r x_0)\sin(k_u z) \\
&\quad -B_{xs}\frac{k_r}{k_u}\cos(k_s y_0)\sinh(k_r x_0)\cos(k_u z) \\
&\quad -B_{yc}\frac{k_s}{k_u}\sin(k_s x_0)\sinh(k_r y_0)\sin(k_u z+\delta) \\
&\quad +B_{ys}\frac{k_s}{k_u}\sin(k_s x_0)\cosh(k_r y_0)\cos(k_u z+\delta)] \qquad (97)
\end{aligned}
$$

$$v_z = v_{z0} \qquad (98)$$

In order to simplify future equations, let

$$K_0 = \frac{qB_0}{mk_u c} \qquad (99)$$

Then

$$
\begin{aligned}
\frac{\gamma}{c}v_x &= -K_0[-\frac{k_s}{k_u}\cos\left(k_u\frac{\Delta_{13}}{2}\right)\sin(k_s y_0)\sinh(k_r x_0)\sin(k_u z) \\
&\quad +\frac{k_s}{k_u}\sin\left(k_u\frac{\Delta_{13}}{2}\right)\sin(k_s y_0)\cosh(k_r x_0)\cos(k_u z) \\
&\quad +\frac{k_r}{k_u}\cos\left(k_u\frac{\Delta_{24}}{2}\right)\cos(k_s x_0)\cosh(k_r y_0)\sin(k_u z+\delta) \\
&\quad -\frac{k_r}{k_u}\sin\left(k_u\frac{\Delta_{24}}{2}\right)\cos(k_s x_0)\sinh(k_r y_0)\cos(k_u z+\delta)] \qquad (100)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\gamma}{c}v_y &= K_0[\frac{k_r}{k_u}\cos\left(k_u\frac{\Delta_{13}}{2}\right)\cos(k_s y_0)\cosh(k_r x_0)\sin(k_u z) \\
&\quad -\frac{k_r}{k_u}\sin\left(k_u\frac{\Delta_{13}}{2}\right)\cos(k_s y_0)\sinh(k_r x_0)\cos(k_u z) \\
&\quad -\frac{k_s}{k_u}\cos\left(k_u\frac{\Delta_{24}}{2}\right)\sin(k_s x_0)\sinh(k_r y_0)\sin(k_u z+\delta) \\
&\quad +\frac{k_s}{k_u}\sin\left(k_u\frac{\Delta_{24}}{2}\right)\sin(k_s x_0)\cosh(k_r y_0)\cos(k_u z+\delta)] \qquad (101)
\end{aligned}
$$

# 4 K Value

## 4.1 Slippage

Slippage is the distance between the light wave and the electrons. The slippage is given by

$$
\begin{aligned}
dS &= (c - v_z)dt \qquad (102) \\
&= (\frac{c}{v_z} - 1)dz \qquad (103)
\end{aligned}
$$

We take the electron energy to be constant, so the Lorentz factor $\gamma$ is constant.

$$\frac{1}{\gamma^2} = 1 - \frac{v_x^2}{c^2} - \frac{v_y^2}{c^2} - \frac{v_z^2}{c^2} \qquad (104)$$

So

$$\frac{v_z^2}{c^2} = 1 - \frac{1}{\gamma^2} - \frac{v_x^2}{c^2} - \frac{v_y^2}{c^2} \tag{105}$$

Taking the square root, we find for large $\gamma$ and small $v_x/c$ and small $v_y/c$

$$\frac{v_z}{c} = 1 - \frac{1}{2\gamma^2} - \frac{v_x^2}{2c^2} - \frac{v_y^2}{2c^2} \tag{106}$$

Taking the inverse, we find

$$\frac{c}{v_z} = 1 + \frac{1}{2\gamma^2} + \frac{v_x^2}{2c^2} + \frac{v_y^2}{2c^2} \tag{107}$$

So the slippage changes with $z$ according to

$$\frac{dS}{dz} = \frac{1}{2\gamma^2} + \frac{v_x^2}{2c^2} + \frac{v_y^2}{2c^2} \tag{108}$$

## 4.2 Define $K$

The slippage changes linearly with $z$ with additional oscillatory terms. The linear term defines the $K$ value.

$$\left\langle \frac{dS}{dz} \right\rangle = \frac{1}{2\gamma^2}\left(1 + \frac{1}{2}K^2\right) \tag{109}$$

So

$$K^2 = 2\left(2\gamma^2 \left\langle \frac{dS}{dz} \right\rangle - 1\right) \tag{110}$$

$$= 2\frac{\gamma^2}{c^2}\left(\langle v_x^2 \rangle + \langle v_y^2 \rangle\right) \tag{111}$$

We know $\frac{\gamma}{c}v_x$ and $\frac{\gamma}{c}v_y$. We need to square them and find the values averaged in $z$.

## 4.3 Calculate $\left\langle \frac{\gamma^2}{c^2} v_x^2 \right\rangle$

From equation 100, the formula for $\frac{\gamma}{c}v_x$ is

$$\begin{aligned} \frac{\gamma}{c}v_x = & -K_0[-\frac{k_s}{k_u}\cos\left(k_u\frac{\Delta_{13}}{2}\right)\sin(k_s y_0)\sinh\left(k_r x_0\right)\sin\left(k_u z\right) \\ & +\frac{k_s}{k_u}\sin\left(k_u\frac{\Delta_{13}}{2}\right)\sin(k_s y_0)\cosh\left(k_r x_0\right)\cos\left(k_u z\right) \\ & +\frac{k_r}{k_u}\cos\left(k_u\frac{\Delta_{24}}{2}\right)\cos(k_s x_0)\cosh\left(k_r y_0\right)\sin\left(k_u z + \delta\right) \\ & -\frac{k_r}{k_u}\sin\left(k_u\frac{\Delta_{24}}{2}\right)\cos(k_s x_0)\sinh\left(k_r y_0\right)\cos\left(k_u z + \delta\right)] \end{aligned} \tag{112}$$

In order to simplify the calculations, we make the following definitions for $c_1$ to $c_4$.

$$\frac{\gamma}{c}v_x = c_1\sin\left(k_u z\right) + c_2\cos\left(k_u z\right) + c_3\sin\left(k_u z + \delta\right) + c_4\cos\left(k_u z + \delta\right) \tag{113}$$

Explicitly, we have

$$
\begin{aligned}
c_1 &= K_0 \frac{k_s}{k_u} \cos\left(k_u \frac{\Delta_{13}}{2}\right) \sin(k_s y_0) \sinh(k_r x_0) && (114)\\
c_2 &= -K_0 \frac{k_s}{k_u} \sin\left(k_u \frac{\Delta_{13}}{2}\right) \sin(k_s y_0) \cosh(k_r x_0) && (115)\\
c_3 &= -K_0 \frac{k_r}{k_u} \cos\left(k_u \frac{\Delta_{24}}{2}\right) \cos(k_s x_0) \cosh(k_r y_0) && (116)\\
c_4 &= K_0 \frac{k_r}{k_u} \sin\left(k_u \frac{\Delta_{24}}{2}\right) \cos(k_s x_0) \sinh(k_r y_0) && (117)
\end{aligned}
$$

Squaring $\frac{\gamma}{c} v_x$, we have

$$
\begin{aligned}
\frac{\gamma^2}{c^2} v_x^2 = \; & c_1^2 \sin^2(k_u z) + c_2^2 \cos^2(k_u z) + c_3^2 \sin^2(k_u z + \delta) + c_4^2 \cos^2(k_u z + \delta) \\
& + 2 c_1 c_2 \sin(k_u z) \cos(k_u z) \\
& + 2 c_1 c_3 \sin(k_u z) \sin(k_u z + \delta) \\
& + 2 c_1 c_4 \sin(k_u z) \cos(k_u z + \delta) \\
& + 2 c_2 c_3 \cos(k_u z) \sin(k_u z + \delta) \\
& + 2 c_2 c_4 \cos(k_u z) \cos(k_u z + \delta) \\
& + 2 c_3 c_4 \sin(k_u z + \delta) \cos(k_u z + \delta) \qquad (118)
\end{aligned}
$$

Taking the average value along $z$, we have

$$
\begin{aligned}
\frac{\gamma^2}{c^2} \left\langle v_x^2 \right\rangle = \; & \frac{1}{2} \left(c_1^2 + c_2^2 + c_3^2 + c_4^2\right) \\
& + 0 \\
& + c_1 c_3 \cos(\delta) \\
& - c_1 c_4 \sin(\delta) \\
& + c_2 c_3 \sin(\delta) \\
& + c_2 c_4 \cos(\delta) \\
& + 0 \qquad (119)
\end{aligned}
$$

### 4.4 Calculate $\left\langle \frac{\gamma^2}{c^2} v_y^2 \right\rangle$

From equation 101, the formula for $\frac{\gamma}{c} v_y$ is

$$
\begin{aligned}
\frac{\gamma}{c} v_y = \; & K_0 [\frac{k_r}{k_u} \cos\left(k_u \frac{\Delta_{13}}{2}\right) \cos(k_s y_0) \cosh(k_r x_0) \sin(k_u z) \\
& - \frac{k_r}{k_u} \sin\left(k_u \frac{\Delta_{13}}{2}\right) \cos(k_s y_0) \sinh(k_r x_0) \cos(k_u z) \\
& - \frac{k_s}{k_u} \cos\left(k_u \frac{\Delta_{24}}{2}\right) \sin(k_s x_0) \sinh(k_r y_0) \sin(k_u z + \delta) \\
& + \frac{k_s}{k_u} \sin\left(k_u \frac{\Delta_{24}}{2}\right) \sin(k_s x_0) \cosh(k_r y_0) \cos(k_u z + \delta)] \qquad (120)
\end{aligned}
$$

We make the following definitions for $d_1$ to $d_4$.

$$v_y = d_1 \sin(k_u z) + d_2 \cos(k_u z) + d_3 \sin(k_u z + \delta) + d_4 \cos(k_u z + \delta)$$

Explicitly, we have

$$\begin{aligned}
d_1 &= K_0 \frac{k_r}{k_u} \cos\left(k_u \frac{\Delta_{13}}{2}\right) \cos(k_s y_0) \cosh(k_r x_0) && (121)\\
d_2 &= -K_0 \frac{k_r}{k_u} \sin\left(k_u \frac{\Delta_{13}}{2}\right) \cos(k_s y_0) \sinh(k_r x_0) && (122)\\
d_3 &= -K_0 \frac{k_s}{k_u} \cos\left(k_u \frac{\Delta_{24}}{2}\right) \sin(k_s x_0) \sinh(k_r y_0) && (123)\\
d_4 &= K_0 \frac{k_s}{k_u} \sin\left(k_u \frac{\Delta_{24}}{2}\right) \sin(k_s x_0) \cosh(k_r y_0) && (124)
\end{aligned}$$

Squaring $\frac{\gamma}{c} v_y$, we have

$$\begin{aligned}
\frac{\gamma^2}{c^2} v_y^2 = \; & d_1^2 \sin^2(k_u z) + d_2^2 \cos^2(k_u z) + d_3^2 \sin^2(k_u z + \delta) + d_4^2 \cos^2(k_u z + \delta)\\
& +2 d_1 d_2 \sin(k_u z) \cos(k_u z)\\
& +2 d_1 d_3 \sin(k_u z) \sin(k_u z + \delta)\\
& +2 d_1 d_4 \sin(k_u z) \cos(k_u z + \delta)\\
& +2 d_2 d_3 \cos(k_u z) \sin(k_u z + \delta)\\
& +2 d_2 d \cos(k_u z) \cos(k_u z + \delta)\\
& +2 d_3 d_4 \sin(k_u z + \delta) \cos(k_u z + \delta) && (125)
\end{aligned}$$

Taking the average value along $z$, we have

$$\begin{aligned}
\frac{\gamma^2}{c^2} \langle v_y^2 \rangle = \; & \frac{1}{2}\left(d_1^2 + d_2^2 + d_3^2 + d_4^2\right)\\
& +0\\
& +d_1 d_3 \cos(\delta)\\
& -d_1 d_4 \sin(\delta)\\
& +d_2 d_3 \sin(\delta)\\
& +d_2 d_4 \cos(\delta)\\
& +0 && (126)
\end{aligned}$$

## 4.5 Calculate $K^2$

### 4.5.1 General Expression

Since

$$K^2 = 2\frac{\gamma^2}{c^2}\left(\langle v_x^2 \rangle + \langle v_y^2 \rangle\right)$$

we can use equations 119 and 126 to write an expression for $K^2$

$$\begin{aligned}
K^2 = \; & \left(c_1^2 + c_2^2 + c_3^2 + c_4^2\right) + \left(d_1^2 + d_2^2 + d_3^2 + d_4^2\right)\\
& +2(c_1 c_3 + c_2 c_4 + d_1 d_3 + d_2 d_4) \cos(\delta)\\
& +2(-c_1 c_4 + c_2 c_3 - d_1 d_4 + d_2 d_3) \sin(\delta) && (127)
\end{aligned}$$

Recalling from equations 114 to 117 and 121 to 124, the $c_i$ and $d_i$ in this expression for $K^2$ have values

$$\begin{aligned}
c_1 &= K_0 \frac{k_s}{k_u} \cos\left(k_u \frac{\Delta_{13}}{2}\right) \sin(k_s y_0) \sinh(k_r x_0) && (128)\\
c_2 &= -K_0 \frac{k_s}{k_u} \sin\left(k_u \frac{\Delta_{13}}{2}\right) \sin(k_s y_0) \cosh(k_r x_0) && (129)\\
c_3 &= -K_0 \frac{k_r}{k_u} \cos\left(k_u \frac{\Delta_{24}}{2}\right) \cos(k_s x_0) \cosh(k_r y_0) && (130)\\
c_4 &= K_0 \frac{k_r}{k_u} \sin\left(k_u \frac{\Delta_{24}}{2}\right) \cos(k_s x_0) \sinh(k_r y_0) && (131)
\end{aligned}$$

$$\begin{aligned}
d_1 &= K_0 \frac{k_r}{k_u} \cos\left(k_u \frac{\Delta_{13}}{2}\right) \cos(k_s y_0) \cosh(k_r x_0) && (132)\\
d_2 &= -K_0 \frac{k_r}{k_u} \sin\left(k_u \frac{\Delta_{13}}{2}\right) \cos(k_s y_0) \sinh(k_r x_0) && (133)\\
d_3 &= -K_0 \frac{k_s}{k_u} \cos\left(k_u \frac{\Delta_{24}}{2}\right) \sin(k_s x_0) \sinh(k_r y_0) && (134)\\
d_4 &= K_0 \frac{k_s}{k_u} \sin\left(k_u \frac{\Delta_{24}}{2}\right) \sin(k_s x_0) \cosh(k_r y_0) && (135)
\end{aligned}$$

These are the relations we seek. With these relations, we know $K$ as a function of position for all polarization modes and row settings of the undulator.

### 4.5.2 Special Case 1, $k_s = 0$

Consider the special case of two crossed planar undulators without field rolloff. In this case $k_s = 0$ and $k_r = k_u$. The $c$ and $d$ parameters in equation 127 then become

$$\begin{aligned}
c_1 &= 0 && (136)\\
c_2 &= 0 && (137)\\
c_3 &= -K_0 \cos\left(k_u \frac{\Delta_{24}}{2}\right) \cosh(k_u y_0) && (138)\\
c_4 &= K_0 \sin\left(k_u \frac{\Delta_{24}}{2}\right) \sinh(k_u y_0) && (139)
\end{aligned}$$

$$\begin{aligned}
d_1 &= K_0 \cos\left(k_u \frac{\Delta_{13}}{2}\right) \cosh(k_u x_0) && (140)\\
d_2 &= -K_0 \sin\left(k_u \frac{\Delta_{13}}{2}\right) \sinh(k_u x_0) && (141)\\
d_3 &= 0 && (142)\\
d_4 &= 0 && (143)
\end{aligned}$$

Inserting these values in the expression for $K^2$ we find

$$K^2 = c_3^2 + c_4^2 + d_1^2 + d_2^2 \qquad (144)$$

$$
\begin{aligned}
K^2 &= K_0^2[\cos^2\left(k_u\frac{\Delta_{24}}{2}\right)\cosh^2(k_u y_0) + \sin^2\left(k_u\frac{\Delta_{24}}{2}\right)\sinh^2(k_u y_0) \\
&\quad + \cos^2\left(k_u\frac{\Delta_{13}}{2}\right)\cosh^2(k_u x_0) + \sin^2\left(k_u\frac{\Delta_{13}}{2}\right)\sinh^2(k_u x_0)] \qquad (145)
\end{aligned}
$$

Note that making the field rolloff small greatly simplifies the formula for $K^2$. The position dependence is second order near the magnetic center for all quadrant row positions. The polarization mode does not enter the equation. It is very desirable to shape the magnets so that $k_s$ is small.

#### 4.5.3 Special Case 2, $x_0 = 0, y_0 = 0$

Consider the special case of being on the magnetic axis where $x_0 = 0$ and $y_0 = 0$. In this case, equations 128 to 135 become

$$
\begin{aligned}
c_1 &= 0 \qquad (146) \\
c_2 &= 0 \qquad (147) \\
c_3 &= -K_0\frac{k_r}{k_u}\cos\left(k_u\frac{\Delta_{24}}{2}\right) \qquad (148) \\
c_4 &= 0 \qquad (149)
\end{aligned}
$$

$$
\begin{aligned}
d_1 &= K_0\frac{k_r}{k_u}\cos\left(k_u\frac{\Delta_{13}}{2}\right) \qquad (150) \\
d_2 &= 0 \qquad (151) \\
d_3 &= 0 \qquad (152) \\
d_4 &= 0 \qquad (153)
\end{aligned}
$$

Inserting these values in equation 127 for $K^2$, we find

$$
\begin{aligned}
K^2 &= c_3^2 + d_1^2 \\
&= K_0^2\left(\frac{k_r}{k_u}\right)^2\left[\cos^2\left(k_u\frac{\Delta_{13}}{2}\right) + \cos^2\left(k_u\frac{\Delta_{24}}{2}\right)\right] \qquad (154)
\end{aligned}
$$

This equation shows that regardless of the value of $k_s$, on the magnetic axis, the undulator has the ideal behavior where the $K$ value is set by shifting the rows of the two crossed undulators. This is true for all polarization modes.

#### 4.5.4 Special Case 3, $\Delta_{13} = \frac{\lambda_u}{2}$, $\Delta_{24} = \frac{\lambda_u}{2}$

This is the case when the undulator is "turned off", that is, on the magnetic axis $K = 0$. In this case,

$$
\begin{aligned}
k_u\frac{\Delta_{13}}{2} &= \frac{\pi}{2} \qquad (155) \\
k_u\frac{\Delta_{24}}{2} &= \frac{\pi}{2} \qquad (156)
\end{aligned}
$$

At general positions we have using equations 128 to 135

$$
\begin{aligned}
c_1 &= 0 \qquad (157) \\
c_2 &= -K_0\frac{k_s}{k_u}\sin(k_s y_0)\cosh(k_r x_0) \qquad (158) \\
c_3 &= 0 \qquad (159) \\
c_4 &= K_0\frac{k_r}{k_u}\cos(k_s x_0)\sinh(k_r y_0) \qquad (160)
\end{aligned}
$$

$$d_1 = 0 \tag{161}$$

$$d_2 = -K_0 \frac{k_r}{k_u} \cos(k_s y_0) \sinh(k_r x_0) \tag{162}$$

$$d_3 = 0 \tag{163}$$

$$d_4 = K_0 \frac{k_s}{k_u} \sin(k_s x_0) \cosh(k_r y_0) \tag{164}$$

Inserting these values in equation 127 for $K^2$, we have

$$\begin{aligned} K^2 &= c_2^2 + c_4^2 + d_2^2 + d_4^2 \\ &\quad +2(c_2 c_4 + d_2 d_4) \cos(\delta) \end{aligned} \tag{165}$$

$$\begin{aligned} K^2 &= K_0^2 \{ \left(\frac{k_s}{k_u}\right)^2 \sin^2(k_s y_0) \cosh^2(k_r x_0) + \left(\frac{k_r}{k_u}\right)^2 \cos^2(k_s x_0) \sinh^2(k_r y_0) \\ &\quad + \left(\frac{k_r}{k_u}\right)^2 \cos^2(k_s y_0) \sinh^2(k_r x_0) + \left(\frac{k_s}{k_u}\right)^2 \sin^2(k_s x_0) \cosh^2(k_r y_0) \\ &\quad +2 \left[ \begin{array}{l} -\frac{k_s k_r}{k_u^2} \sin(k_s y_0) \cosh(k_r x_0) \cos(k_s x_0) \sinh(k_r y_0) \\ -\frac{k_s k_r}{k_u^2} \cos(k_s y_0) \sinh(k_r x_0) \sin(k_s x_0) \cosh(k_r y_0) \end{array} \right] \cos(\delta) \} \end{aligned} \tag{166}$$

Note that away from the magnetic axis, the undulator does not "turn off". It still has a finite $K$ value. For the special case of $k_s = 0$,

$$K^2 = K_0^2 \left[\sinh^2(k_u y_0) + \sinh^2(k_u x_0)\right] \tag{167}$$

The undulator does not have $K = 0$ off axis even if $k_s = 0$.

# 5 K Values Near The Magnetic Center

Typically the undulator is aligned so that the beam is close to the magnetic center. In this case we expand the $c$ and $d$ parameters of equations 128 to 135 to second order in $k_r x_0$, $k_s x_0$, $k_r y_0$ and $k_s y_0$. We get the following results.

$$c_1 = K_0 \frac{k_s^2 k_r}{k_u} x_0 y_0 \cos\left(k_u \frac{\Delta_{13}}{2}\right) \tag{168}$$

$$c_2 = -K_0 \frac{k_s^2}{k_u} y_0 \sin\left(k_u \frac{\Delta_{13}}{2}\right) \tag{169}$$

$$c_3 = -K_0 \frac{k_r}{k_u} \left(1 - \frac{1}{2} k_s^2 x_0^2 + \frac{1}{2} k_r^2 y_0^2\right) \cos\left(k_u \frac{\Delta_{24}}{2}\right) \tag{170}$$

$$c_4 = K_0 \frac{k_r^2}{k_u} y_0 \sin\left(k_u \frac{\Delta_{24}}{2}\right) \tag{171}$$

$$
\begin{aligned}
d_1 &= K_0 \frac{k_r}{k_u} \left(1 - \frac{1}{2} k_s^2 y_0^2 + \frac{1}{2} k_r^2 x_0^2\right) \cos\left(k_u \frac{\Delta_{13}}{2}\right) && (172)\\
d_2 &= -K_0 \frac{k_r^2}{k_u} x_0 \sin\left(k_u \frac{\Delta_{13}}{2}\right) && (173)\\
d_3 &= -K_0 \frac{k_s^2 k_r}{k_u} x_0 y_0 \cos\left(k_u \frac{\Delta_{24}}{2}\right) && (174)\\
d_4 &= K_0 \frac{k_s^2}{k_u} x_0 \sin\left(k_u \frac{\Delta_{24}}{2}\right) && (175)
\end{aligned}
$$

We can now find $K^2$ to second order in $k_r x_0$, $k_s x_0$, $k_r y_0$ and $k_s y_0$. Equation 127 gives

$$
\begin{aligned}
K^2 = &\left(c_1^2 + c_2^2 + c_3^2 + c_4^2\right) + \left(d_1^2 + d_2^2 + d_3^2 + d_4^2\right)\\
&+2(c_1 c_3 + c_2 c_4 + d_1 d_3 + d_2 d_4) \cos(\delta)\\
&+2(-c_1 c_4 + c_2 c_3 - d_1 d_4 + d_2 d_3) \sin(\delta) \qquad (176)
\end{aligned}
$$

Inserting the second order position dependence, we get

$$
\begin{aligned}
K^2 = & \left[-K_0 \frac{k_s^2}{k_u} y_0 \sin\left(k_u \frac{\Delta_{13}}{2}\right)\right]^2 + \left[-K_0 \frac{k_r}{k_u} \left(1 - \frac{1}{2} k_s^2 x_0^2 + \frac{1}{2} k_r^2 y_0^2\right) \cos\left(k_u \frac{\Delta_{24}}{2}\right)\right]^2\\
&+ \left[K_0 \frac{k_r^2}{k_u} y_0 \sin\left(k_u \frac{\Delta_{24}}{2}\right)\right]^2 + \left[K_0 \frac{k_r}{k_u} \left(1 - \frac{1}{2} k_s^2 y_0^2 + \frac{1}{2} k_r^2 x_0^2\right) \cos\left(k_u \frac{\Delta_{13}}{2}\right)\right]^2\\
&+ \left[-K_0 \frac{k_r^2}{k_u} x_0 \sin\left(k_u \frac{\Delta_{13}}{2}\right)\right]^2 + \left[K_0 \frac{k_s^2}{k_u} x_0 \sin\left(k_u \frac{\Delta_{24}}{2}\right)\right]^2\\
&+2 \left\{ \begin{array}{c}
\left[K_0 \frac{k_s^2 k_r}{k_u} x_0 y_0 \cos\left(k_u \frac{\Delta_{13}}{2}\right)\right] \left[-K_0 \frac{k_r}{k_u} \cos\left(k_u \frac{\Delta_{24}}{2}\right)\right]\\
+ \left[-K_0 \frac{k_s^2}{k_u} y_0 \sin\left(k_u \frac{\Delta_{13}}{2}\right)\right] \left[K_0 \frac{k_r^2}{k_u} y_0 \sin\left(k_u \frac{\Delta_{24}}{2}\right)\right]\\
+ \left[K_0 \frac{k_r}{k_u} \cos\left(k_u \frac{\Delta_{13}}{2}\right)\right] \left[-K_0 \frac{k_s^2 k_r}{k_u} x_0 y_0 \cos\left(k_u \frac{\Delta_{24}}{2}\right)\right]\\
+ \left[-K_0 \frac{k_r^2}{k_u} x_0 \sin\left(k_u \frac{\Delta_{13}}{2}\right)\right] \left[K_0 \frac{k_s^2}{k_u} x_0 \sin\left(k_u \frac{\Delta_{24}}{2}\right)\right]
\end{array} \right\} \cos(\delta)\\
&+2 \left\{ \begin{array}{c}
\left[-K_0 \frac{k_s^2}{k_u} y_0 \sin\left(k_u \frac{\Delta_{13}}{2}\right)\right] \left[-K_0 \frac{k_r}{k_u} \cos\left(k_u \frac{\Delta_{24}}{2}\right)\right]\\
- \left[K_0 \frac{k_r}{k_u} \cos\left(k_u \frac{\Delta_{13}}{2}\right)\right] \left[K_0 \frac{k_s^2}{k_u} x_0 \sin\left(k_u \frac{\Delta_{24}}{2}\right)\right]
\end{array} \right\} \sin(\delta) \qquad (177)
\end{aligned}
$$

We proceed to simplify this expression.

$$
\begin{aligned}
K^2 = & \frac{K_0^2}{k_u^2} k_s^4 y_0^2 \sin^2\left(k_u \frac{\Delta_{13}}{2}\right) + \frac{K_0^2}{k_u^2} k_r^2 \left(1 - k_s^2 x_0^2 + k_r^2 y_0^2\right) \cos^2\left(k_u \frac{\Delta_{24}}{2}\right)\\
&+ \frac{K_0^2}{k_u^2} k_r^4 y_0^2 \sin^2\left(k_u \frac{\Delta_{24}}{2}\right) + \frac{K_0^2}{k_u^2} k_r^2 \left(1 - k_s^2 y_0^2 + k_r^2 x_0^2\right) \cos^2\left(k_u \frac{\Delta_{13}}{2}\right)\\
&+ \frac{K_0^2}{k_u^2} k_r^4 x_0^2 \sin^2\left(k_u \frac{\Delta_{13}}{2}\right) + \frac{K_0^2}{k_u^2} k_s^4 x_0^2 \sin^2\left(k_u \frac{\Delta_{24}}{2}\right)\\
&+2 \left\{ \begin{array}{c}
-\frac{K_0^2}{k_u^2} k_s^2 k_r^2 x_0 y_0 \cos\left(k_u \frac{\Delta_{13}}{2}\right) \cos\left(k_u \frac{\Delta_{24}}{2}\right)\\
-\frac{K_0^2}{k_u^2} k_s^2 k_r^2 y_0^2 \sin\left(k_u \frac{\Delta_{13}}{2}\right) \sin\left(k_u \frac{\Delta_{24}}{2}\right)\\
-\frac{K_0^2}{k_u^2} k_s^2 k_r^2 x_0 y_0 \cos\left(k_u \frac{\Delta_{13}}{2}\right) \cos\left(k_u \frac{\Delta_{24}}{2}\right)\\
-\frac{K_0^2}{k_u^2} k_s^2 k_r^2 x_0^2 \sin\left(k_u \frac{\Delta_{13}}{2}\right) \sin\left(k_u \frac{\Delta_{24}}{2}\right)
\end{array} \right\} \cos(\delta)\\
&+2 \left\{ \begin{array}{c}
\frac{K_0^2}{k_u^2} k_s^2 k_r y_0 \sin\left(k_u \frac{\Delta_{13}}{2}\right) \cos\left(k_u \frac{\Delta_{24}}{2}\right)\\
-\frac{K_0^2}{k_u^2} k_s^2 k_r x_0 \cos\left(k_u \frac{\Delta_{13}}{2}\right) \sin\left(k_u \frac{\Delta_{24}}{2}\right)
\end{array} \right\} \sin(\delta) \qquad (178)
\end{aligned}
$$

Ordering the terms in powers of $k_r x_0$, $k_s x_0$, $k_r y_0$ and $k_s y_0$, we find

$$\begin{aligned} K^2 &= \frac{K_0^2}{k_u^2}\{ \\ & k_r^2\left[\cos^2\left(k_u\frac{\Delta_{13}}{2}\right)+\cos^2\left(k_u\frac{\Delta_{24}}{2}\right)\right] \\ & +2k_s^2k_r\left[\begin{array}{c} y_0\sin\left(k_u\frac{\Delta_{13}}{2}\right)\cos\left(k_u\frac{\Delta_{24}}{2}\right) \\ -x_0\cos\left(k_u\frac{\Delta_{13}}{2}\right)\sin\left(k_u\frac{\Delta_{24}}{2}\right)\end{array}\right]\sin(\delta) \\ & +k_s^4y_0^2\sin^2\left(k_u\frac{\Delta_{13}}{2}\right)+k_r^2k_s^2x_0^2\cos^2\left(k_u\frac{\Delta_{24}}{2}\right) \\ & +k_r^4y_0^2\cos^2\left(k_u\frac{\Delta_{24}}{2}\right)+k_r^4y_0^2\sin^2\left(k_u\frac{\Delta_{24}}{2}\right) \\ & +k_r^2k_s^2y_0^2\cos^2\left(k_u\frac{\Delta_{13}}{2}\right)+k_r^4x_0^2\cos^2\left(k_u\frac{\Delta_{13}}{2}\right) \\ & +k_r^4x_0^2\sin^2\left(k_u\frac{\Delta_{13}}{2}\right)+k_s^4x_0^2\sin^2\left(k_u\frac{\Delta_{24}}{2}\right) \\ & -2k_s^2k_r^2\left[\begin{array}{c} x_0y_0\cos\left(k_u\frac{\Delta_{13}}{2}\right)\cos\left(k_u\frac{\Delta_{24}}{2}\right) \\ +y_0^2\sin\left(k_u\frac{\Delta_{13}}{2}\right)\sin\left(k_u\frac{\Delta_{24}}{2}\right) \\ +x_0y_0\cos\left(k_u\frac{\Delta_{13}}{2}\right)\cos\left(k_u\frac{\Delta_{24}}{2}\right) \\ +x_0^2\sin\left(k_u\frac{\Delta_{13}}{2}\right)\sin\left(k_u\frac{\Delta_{24}}{2}\right)\end{array}\right]\cos(\delta)\} \end{aligned} \tag{179}$$

Rearranging the second order terms and simplifying further, we find

$$\begin{aligned} K^2 &= \frac{K_0^2}{k_u^2}\{ \\ & k_r^2\left[\cos^2\left(k_u\frac{\Delta_{13}}{2}\right)+\cos^2\left(k_u\frac{\Delta_{24}}{2}\right)\right] \\ & +2k_s^2k_r\left[\begin{array}{c} y_0\sin\left(k_u\frac{\Delta_{13}}{2}\right)\cos\left(k_u\frac{\Delta_{24}}{2}\right) \\ -x_0\cos\left(k_u\frac{\Delta_{13}}{2}\right)\sin\left(k_u\frac{\Delta_{24}}{2}\right)\end{array}\right]\sin(\delta) \\ & +\left(k_s^4y_0^2+k_r^4x_0^2\right)\sin^2\left(k_u\frac{\Delta_{13}}{2}\right)+k_r^2\left(k_r^2x_0^2+k_s^2y_0^2\right)\cos^2\left(k_u\frac{\Delta_{13}}{2}\right) \\ & +\left(k_s^4x_0^2+k_r^4y_0^2\right)\sin^2\left(k_u\frac{\Delta_{24}}{2}\right)+k_r^2\left(k_s^2x_0^2+k_r^2y_0^2\right)\cos^2\left(k_u\frac{\Delta_{24}}{2}\right) \\ & -2k_s^2k_r^2\left[\begin{array}{c} 2x_0y_0\cos\left(k_u\frac{\Delta_{13}}{2}\right)\cos\left(k_u\frac{\Delta_{24}}{2}\right) \\ +\left(x_0^2+y_0^2\right)\sin\left(k_u\frac{\Delta_{13}}{2}\right)\sin\left(k_u\frac{\Delta_{24}}{2}\right)\end{array}\right]\cos(\delta)\} \end{aligned} \tag{180}$$

This is our desired result. Note that when $\delta \neq 0$ and $\delta \neq \pi$, there is a first order gradient in the $K$ value.

The gradient of $K^2$ near the magnetic center is problematic for aligning the undulator. To first order in $k_r x_0$, $k_s x_0$, $k_r y_0$ and $k_s y_0$, $K$ can be expressed as

$$K = \frac{K_0}{k_u}\sqrt{k_r^2\left[\cos^2\left(k_u\frac{\Delta_{13}}{2}\right)+\cos^2\left(k_u\frac{\Delta_{24}}{2}\right)\right]}$$

$$\times\left[1+\frac{k_s^2k_r}{k_r^2\left[\cos^2\left(k_u\frac{\Delta_{13}}{2}\right)+\cos^2\left(k_u\frac{\Delta_{24}}{2}\right)\right]}\left[\begin{array}{c} y_0\sin\left(k_u\frac{\Delta_{13}}{2}\right)\cos\left(k_u\frac{\Delta_{24}}{2}\right) \\ -x_0\cos\left(k_u\frac{\Delta_{13}}{2}\right)\sin\left(k_u\frac{\Delta_{24}}{2}\right)\end{array}\right]\sin(\delta)\right] \tag{181}$$

The gradient in $K$ at the magnetic center is given by

$$\frac{\partial K}{\partial x_0} = -\frac{K_0}{k_u} \frac{k_s^2}{\sqrt{\left[\cos^2\left(k_u \frac{\Delta_{13}}{2}\right) + \cos^2\left(k_u \frac{\Delta_{24}}{2}\right)\right]}} \cos\left(k_u \frac{\Delta_{13}}{2}\right) \sin\left(k_u \frac{\Delta_{24}}{2}\right) \sin(\delta) \tag{182}$$

$$\frac{\partial K}{\partial y_0} = \frac{K_0}{k_u} \frac{k_s^2}{\sqrt{\left[\cos^2\left(k_u \frac{\Delta_{13}}{2}\right) + \cos^2\left(k_u \frac{\Delta_{24}}{2}\right)\right]}} \sin\left(k_u \frac{\Delta_{13}}{2}\right) \cos\left(k_u \frac{\Delta_{24}}{2}\right) \sin(\delta) \tag{183}$$

We will calculate the size of these terms below.

# 6 Numerical Calculations

## 6.1 Parameter Values

In the calculations below, the following parameter values are used:

$$\lambda_u = .032 \text{ m} \tag{184}$$

$$k_u = \frac{2\pi}{\lambda_u} = 196 \text{ 1/m} \tag{185}$$

$$k_r = 270 \text{ 1/m} \tag{186}$$

$$k_s = 186 \text{ 1/m} \tag{187}$$

$$B_0 = .61 \text{ T} \tag{188}$$

$$K_0 = 1.82 \tag{189}$$

The value for $k_s$ was measured in the Delta undulator[3]. The value for $k_r$ came from the constraint $k_r^2 = k_s^2 + k_u^2$. The peak quadrant field $B_0$ was calculated from the peak measured field of 1.2 T in linear polarization vertical field mode. The peak field in this mode $B_{peak}$ is given by $4\phi_{0Q}k_r/\sqrt{2}$ and $B_0$ is given by $2\phi_{0Q}k_u$, so $B_0 = B_{peak}\frac{k_u}{\sqrt{2}k_r}$. The value for $K_0$ was calculated from $B_0$ using equation 99.

## 6.2 Primary Modes

From equation 180 we see that the linear terms near the magnetic center are largest when $\delta = \pm\pi/2$ and they are smallest when $\delta = 0$ or $\pi$. We now consider these cases. $\delta = 0$ gives the linear polarization vertical magnetic field mode, $\delta = \pi$ gives the linear polarization horizontal magnetic field mode, $\delta = -\pi/2$ gives the circular polarization right hand magnetic field mode, and $\delta = \pi/2$ gives the circular polarization left hand magnetic field mode. In these primary modes $\Delta_{13} = \Delta_{24}$ and the $K$ value is set by the $\Delta_{13}$ and $\Delta_{24}$ row shifts[4].

### 6.2.1 Linear Polarization Vertical Magnetic Field

Figure 3 shows how the $K$ value changes with $x_0$, the horizontal position in the undulator frame, when $y_0 = 0$. For this plot, the rows were shifted so that $\Delta_{13} = \Delta_{24} = .008$ m, and $\delta = 0$. This is the linear polarization vertical field mode. On the magnetic axis, $K$ is given by

$$K = K_0 \left(\frac{k_r}{k_u}\right) \sqrt{\cos^2\left(k_u \frac{\Delta_{13}}{2}\right) + \cos^2\left(k_u \frac{\Delta_{24}}{2}\right)} \tag{190}$$

[3] Yurii Levashov, private communication.

[4] Z. Wolf, H.-D. Nuhn, "Setting The K Value And Polarization Mode Of The Delta Undulator", LCLS-TN-14-2, September, 2014.

The maximum value is obtained when $\cos\left(k_u \frac{\Delta_{13}}{2}\right) = 1$ and $\cos\left(k_u \frac{\Delta_{24}}{2}\right) = 1$, and is

$$K_{\max} = K_0 \left(\frac{k_r}{k_u}\right) \sqrt{2} \tag{191}$$

When $\Delta_{13} = \Delta_{24} = .008$ m, $\cos\left(k_u \frac{\Delta_{13}}{2}\right) = 1/\sqrt{2}$ and $\cos\left(k_u \frac{\Delta_{24}}{2}\right) = 1/\sqrt{2}$, so

$$K = K_0 \left(\frac{k_r}{k_u}\right) \tag{192}$$

and

$$\frac{K}{K_{\max}} = \frac{1}{\sqrt{2}} \tag{193}$$

Figure 3 has several curves. The blue curve is calculated from equation 127. The green dashed curve is an independent calculation starting with the expressions for the magnetic field components and numerically integrating to get the transverse velocities and from them to calculate $K$. This provides a check on the calculations in this note. The red curve is from equation 180. It gives an approximate expression for $K$ near the magnetic center. The black line is a linear fit to the curves at the magnetic center. It uses equation 190 to find the $K$ value at the magnetic center and equation 182 to find the slope. The values of $K$ at the magnetic center, $K(0,0)$, and the slope at the magnetic center, $dK/dx(0,0)$, are shown in the figure.

The red curve deviates from the blue curve away from $x_0 = 0$, $y_0 = 0$. We expect the second order approximation to be valid for $k_r x_0$, $k_s x_0$, $k_r y_0$ and $k_s y_0$ much less than 1. Since $k_s < k_r$, we expect the approximation to be valid for

$$x_0 \ll \frac{1}{k_s} \tag{194}$$

or

$$x_0 \ll 5 \text{ mm} \tag{195}$$

For $x_0 = 0.5$ mm, $k_s x_0 = 0.1$, $(k_s x_0)^2 = 0.01$, and the missing third order terms should contribute about $K \times 0.001 = .003$ to $K$. This is roughly in line with the deviations of the red line that we see.

We can zoom in on the region near $x_0 = 0$ in order to estimate the alignment tolerance of the undulator. This is done in figure 4. From the blue curve, $K$ changes by about $10^{-4}$ relative to its value at the magnetic center when $x_0$ is approximately 70 microns from the magnetic center. This is in line with the hyperbolic cosine position dependence of a planar undulator, and is in fact very close to the value in the LCLS undulators which have a similar period (30 mm).

The plot of $K$ vs $y_0$ at $x_0 = 0$ is shown in figure 5. The figure is identical to the plot of $K$ vs $x_0$ at $y_0 = 0$, as we expect from the symmetry of the undulator.

### 6.2.2 Linear Polarization Horizontal Magnetic Field

Figure 6 shows how the $K$ value changes with horizontal position $x_0$ when $y_0 = 0$. For this plot, the rows were shifted such that $\Delta_{13} = \Delta_{24} = .008$ m, and $\delta = \pi$. This is the linear polarization horizontal field mode. $K$ changes with position in this mode in the same way as in the linear polarization vertical field mode, as one would expect from symmetry.

### 6.2.3 Circular Polarization Right Hand Magnetic Field

Figure 7 shows how the $K$ value changes with horizontal position $x_0$ when $y_0 = 0$. For this plot, the rows were shifted such that $\Delta_{13} = \Delta_{24} = .008$ m, and $\delta = -\pi/2$. This is the circular polarization right hand field mode. Note the gradient at the magnetic center.

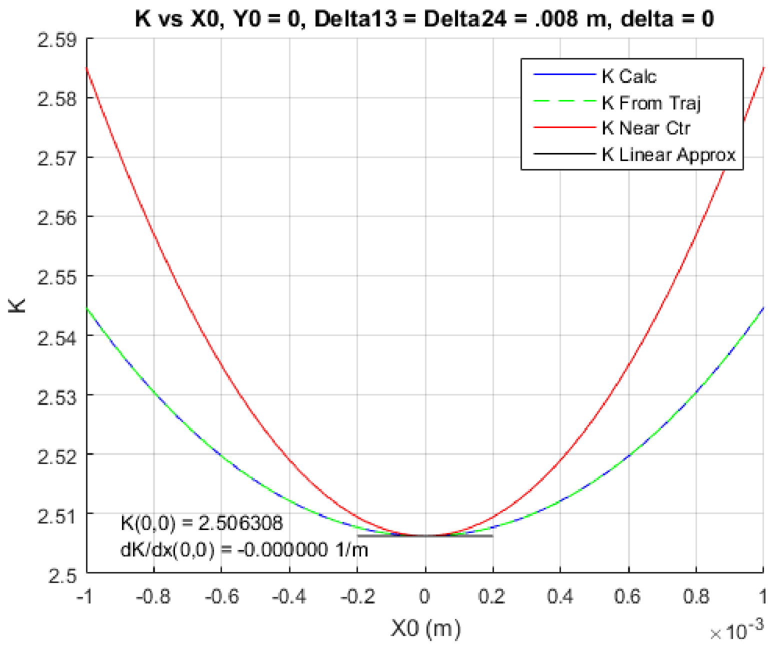


Figure 3: This figure shows how $K$ varies with $x_0$ when $y_0 = 0$ in linear polarization vertical magnetic field mode.

Figure 8 shows how $K$ varies near the magnetic center. The slope of the $K$ change with $x_0$ is very large, 161 1/m. A misalignment of the undulator by 100 microns changes $K$ by 0.016, or 0.6%. This is a very large change for a modest misalignment.

Figure 9 shows how $K$ varies with $y_0$ when $x_0 = 0$. The sign of the change of $K$ with $y_0$ is opposite to the sign of the change of $K$ with $x_0$, but the magnitude is the same, as expected from equation 180 with $\Delta_{13} = \Delta_{24}$.

#### 6.2.4 Circular Polarization Left Hand Magnetic Field

Figure 10 shows how the $K$ value changes with horizontal position $x_0$ when $y_0 = 0$. For this plot, the rows were shifted such that $\Delta_{13} = \Delta_{24} = .008$ m, and $\delta = \pi/2$. This is the circular polarization left hand field mode. Again note the gradient at the magnetic center. The variation of $K$ with position in this mode is very similar to the case of right hand circular polarization, as one would expect from symmetry.

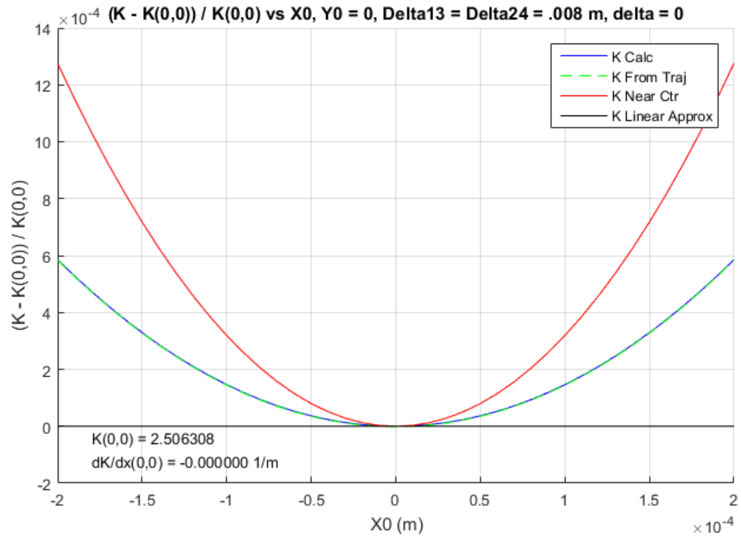


Figure 4: Relative deviation of $K$ from its value at the magnetic center in a small region near the magnetic center.

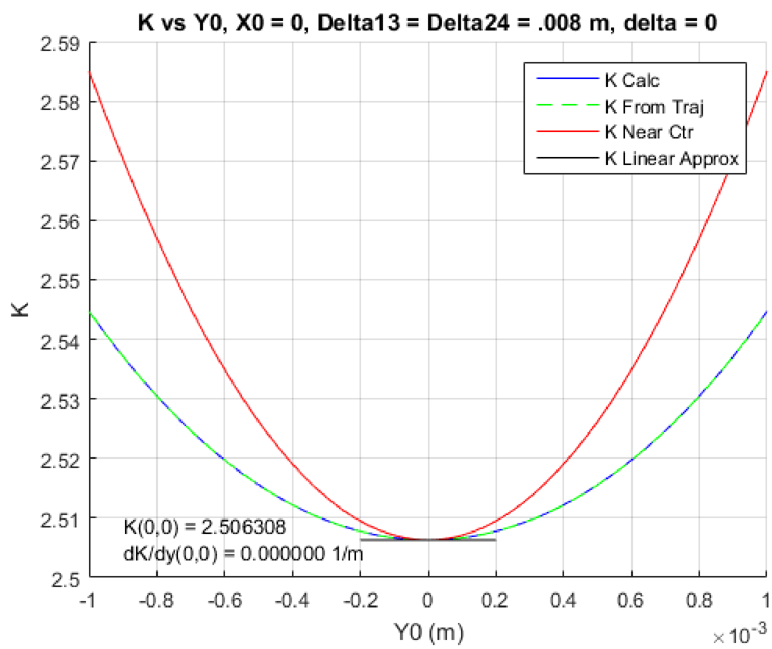


Figure 5: $K$ variation with $y_0$ when $x_0 = 0$ in linear polarization vertical magnetic field mode.

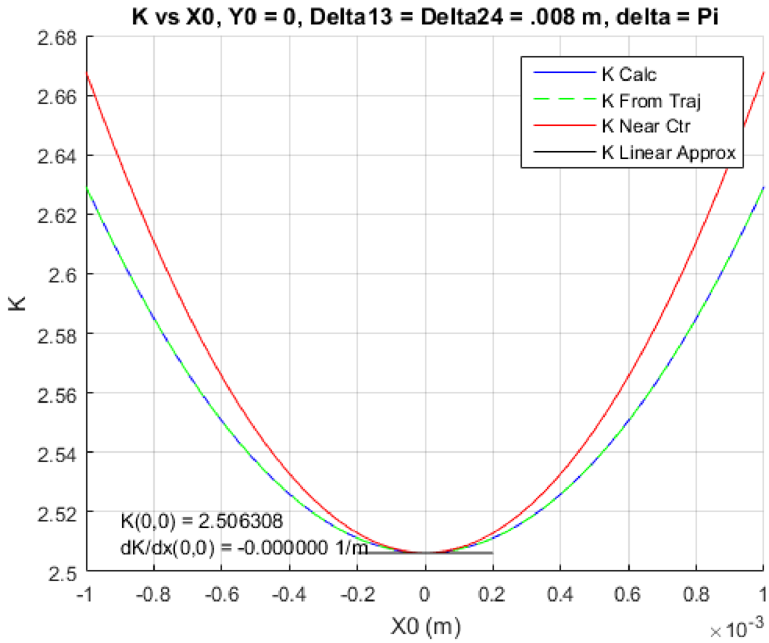


Figure 6: This figure shows how $K$ varies with $x_0$ when $y_0 = 0$ in linear polarization horizontal magnetic field mode.

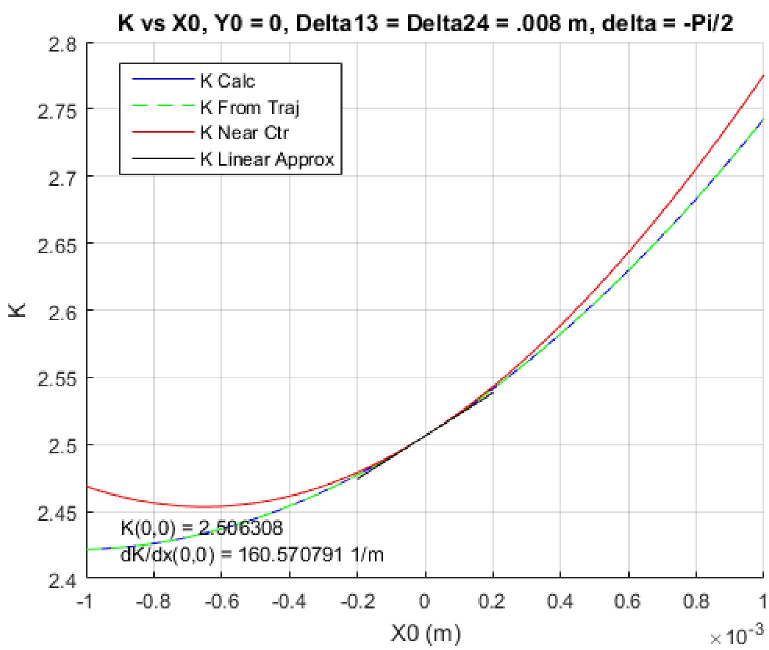


Figure 7: This figure shows how $K$ varies with $x_0$ when $y_0 = 0$ in circular polarization right hand magnetic field mode.

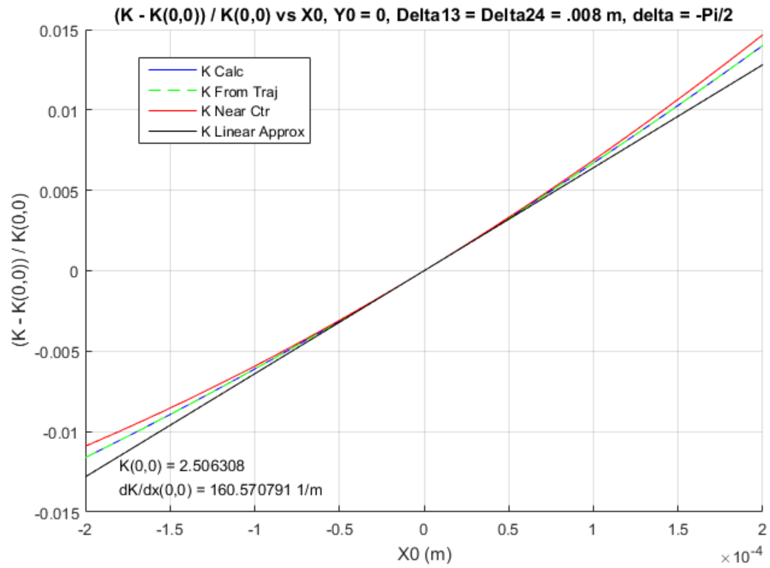


Figure 8: $K$ variation with $x_0$ when $y_0 = 0$ in circular polarization right hand magnetic field mode. It is a zoom view near the magnetic center.

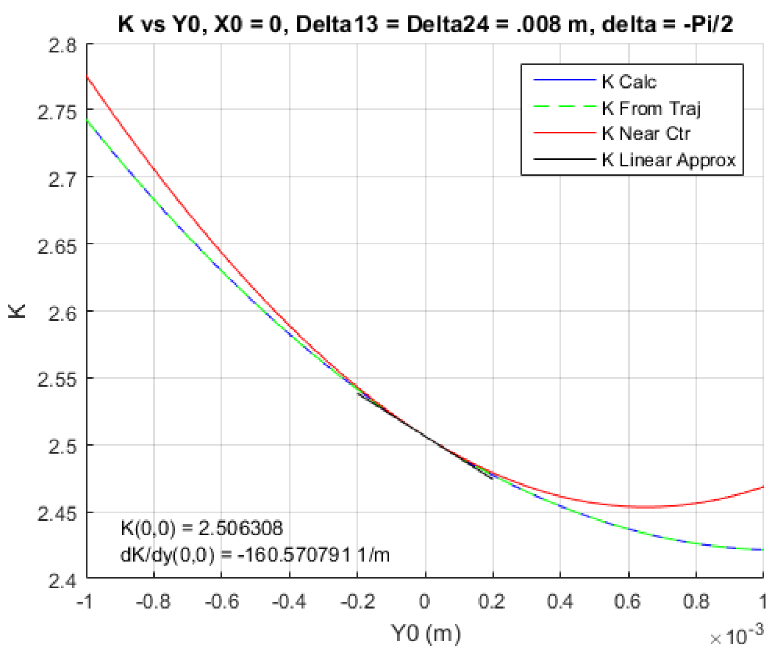


Figure 9: This figure shows how $K$ varies with $y_0$ when $x_0 = 0$ in circular polarization right hand magnetic field mode.

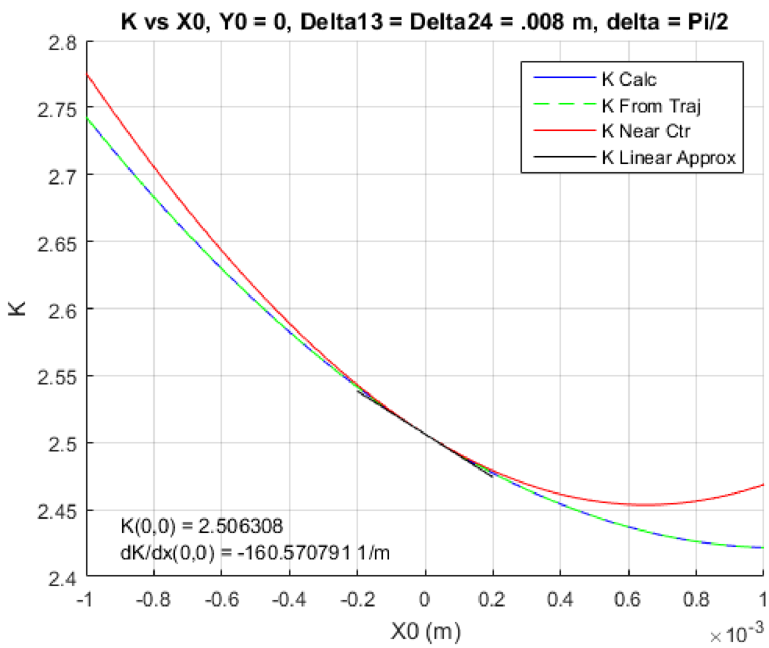


Figure 10: Circular polarization left hand magnetic field mode $K$ vs $x_0$ with $y_0 = 0$.

# 7 Laboratory Coordinate System

The calculations for the Delta undulator are most easily done in the undulator reference frame. This was used in all the calculations up to this point. For comparing the calculations to measurements, however, it is useful to also work in the laboratory reference frame. The relation between frames is

$$x_{0L} = \frac{1}{\sqrt{2}}(x_0 - y_0) \tag{196}$$

$$y_{0L} = \frac{1}{\sqrt{2}}(x_0 + y_0) \tag{197}$$

or equivalently

$$x_0 = \frac{1}{\sqrt{2}}(x_{0L} + y_{0L}) \tag{198}$$

$$y_0 = \frac{1}{\sqrt{2}}(-x_{0L} + y_{0L}) \tag{199}$$

The subscript $L$ indicates the laboratory frame, and the unsubscripted quantities refer to the undulator frame.

Consider equation 181 giving to first order in $k_r x_0$, $k_s x_0$, $k_r y_0$ and $k_s y_0$ the expression for the $K$ value near the magnetic center and reproduced below.

$$K = \frac{K_0}{k_u}\sqrt{k_r^2\left[\cos^2\left(k_u\frac{\Delta_{13}}{2}\right) + \cos^2\left(k_u\frac{\Delta_{24}}{2}\right)\right]}$$
$$\times\left[1 + \frac{k_s^2 k_r}{k_r^2\left[\cos^2\left(k_u\frac{\Delta_{13}}{2}\right) + \cos^2\left(k_u\frac{\Delta_{24}}{2}\right)\right]}\left[\begin{array}{c} y_0\sin\left(k_u\frac{\Delta_{13}}{2}\right)\cos\left(k_u\frac{\Delta_{24}}{2}\right) \\ -x_0\cos\left(k_u\frac{\Delta_{13}}{2}\right)\sin\left(k_u\frac{\Delta_{24}}{2}\right)\end{array}\right]\sin(\delta)\right]$$

Expressing this formula in laboratory coordinates, we find

$$K = \frac{K_0}{k_u}\sqrt{k_r^2\left[\cos^2\left(k_u\frac{\Delta_{13}}{2}\right) + \cos^2\left(k_u\frac{\Delta_{24}}{2}\right)\right]}$$
$$\times\left[1 + \frac{k_s^2 k_r}{k_r^2\left[\cos^2\left(k_u\frac{\Delta_{13}}{2}\right) + \cos^2\left(k_u\frac{\Delta_{24}}{2}\right)\right]}\left[\begin{array}{c} \frac{1}{\sqrt{2}}(-x_{0L} + y_{0L})\sin\left(k_u\frac{\Delta_{13}}{2}\right)\cos\left(k_u\frac{\Delta_{24}}{2}\right) \\ -\frac{1}{\sqrt{2}}(x_{0L} + y_{0L})\cos\left(k_u\frac{\Delta_{13}}{2}\right)\sin\left(k_u\frac{\Delta_{24}}{2}\right)\end{array}\right]\sin(\delta)\right] \tag{200}$$

which can be expanded to give

$$K = \frac{K_0}{k_u}\sqrt{k_r^2\left[\cos^2\left(k_u\frac{\Delta_{13}}{2}\right) + \cos^2\left(k_u\frac{\Delta_{24}}{2}\right)\right]}$$
$$-x_{0L}\{\frac{K_0}{k_u}\frac{k_s^2 k_r}{\sqrt{2k_r^2\left[\cos^2\left(k_u\frac{\Delta_{13}}{2}\right) + \cos^2\left(k_u\frac{\Delta_{24}}{2}\right)\right]}}$$
$$\times\left[\sin\left(k_u\frac{\Delta_{13}}{2}\right)\cos\left(k_u\frac{\Delta_{24}}{2}\right) + \cos\left(k_u\frac{\Delta_{13}}{2}\right)\sin\left(k_u\frac{\Delta_{24}}{2}\right)\right]\sin(\delta)\}$$
$$+y_{0L}\{\frac{K_0}{k_u}\frac{k_s^2 k_r}{\sqrt{2k_r^2\left[\cos^2\left(k_u\frac{\Delta_{13}}{2}\right) + \cos^2\left(k_u\frac{\Delta_{24}}{2}\right)\right]}}$$
$$\times\left[\sin\left(k_u\frac{\Delta_{13}}{2}\right)\cos\left(k_u\frac{\Delta_{24}}{2}\right) - \cos\left(k_u\frac{\Delta_{13}}{2}\right)\sin\left(k_u\frac{\Delta_{24}}{2}\right)\right]\sin(\delta)\} \tag{201}$$

The gradients in $K$ at the magnetic center in the laboratory frame are

$$\frac{\partial K}{\partial x_{0L}} = -\frac{K_0}{k_u}\frac{k_s^2}{\sqrt{2\left[\cos^2\left(k_u\frac{\Delta_{13}}{2}\right)+\cos^2\left(k_u\frac{\Delta_{24}}{2}\right)\right]}} \times\left[\sin\left(k_u\frac{\Delta_{13}}{2}\right)\cos\left(k_u\frac{\Delta_{24}}{2}\right)+\cos\left(k_u\frac{\Delta_{13}}{2}\right)\sin\left(k_u\frac{\Delta_{24}}{2}\right)\right]\sin\left(\delta\right) \qquad (202)$$

$$\frac{\partial K}{\partial y_{0L}} = \frac{K_0}{k_u}\frac{k_s^2}{\sqrt{2\left[\cos^2\left(k_u\frac{\Delta_{13}}{2}\right)+\cos^2\left(k_u\frac{\Delta_{24}}{2}\right)\right]}} \times\left[\sin\left(k_u\frac{\Delta_{13}}{2}\right)\cos\left(k_u\frac{\Delta_{24}}{2}\right)-\cos\left(k_u\frac{\Delta_{13}}{2}\right)\sin\left(k_u\frac{\Delta_{24}}{2}\right)\right]\sin\left(\delta\right) \qquad (203)$$

In the primary modes, $\Delta_{13} = \Delta_{24} = \Delta$. In this case

$$\frac{\partial K}{\partial x_{0L}} = -\frac{K_0}{k_u}k_s^2\sin\left(k_u\frac{\Delta}{2}\right)\sin\left(\delta\right) \qquad (204)$$

$$\frac{\partial K}{\partial y_{0L}} = 0 \qquad (205)$$

In the linear modes, $\sin(\delta) = 0$ and there are no gradients. In the circular modes, however, $\sin(\delta) = \mp 1$ for right and left handed magnetic fields, respectively. In this case for the circular modes

$$\frac{\partial K}{\partial x_{0L}} = \pm\frac{K_0}{k_u}k_s^2\sin\left(k_u\frac{\Delta}{2}\right) \qquad (206)$$

$$\frac{\partial K}{\partial y_{0L}} = 0 \qquad (207)$$

Inserting the parameter values used previously, namely

$$k_u = 196\ 1/\text{m} \qquad (208)$$

$$k_s = 186\ 1/\text{m} \qquad (209)$$

$$K_0 = 1.82 \qquad (210)$$

$$\sin\left(k_u\frac{\Delta}{2}\right) = \frac{1}{\sqrt{2}} \qquad (211)$$

we get

$$\frac{\partial K}{\partial x_{0L}} = \pm 227\ 1/\text{m} \qquad (212)$$

$$\frac{\partial K}{\partial y_{0L}} = 0 \qquad (213)$$

where the plus sign is for right handed magnetic fields and the minus sign is for left handed magnetic fields. This is a very large gradient in $x_{0L}$. A misalignment of 100 microns in $x_L$ causes $K$ to change by .0227 for this $\Delta$ setting. Note that smaller values of $\Delta$ give a smaller gradient. A misalignment in $y_L$ does not produce a change in $K$.

# 8 Conclusion

The general expressions for the position dependence of $K$ were derived in this note. We found that it is highly desirable to have $k_s$ as small as possible. With $k_s > 0$, there are first order gradients in $K$ with position, and the $K$ value will be difficult to set by conventional magnetic measurement and alignment techniques.

**Acknowledgements**
I am grateful to Heinz-Dieter Nuhn for many discussions about this work.