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Beam Trajectories In The Delta Undulator II

Zachary Wolf
SLAC

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Abstract

If the field from the permanent magnets making up the Delta undulator has significant rolloff away from the center line of the magnets, the beam trajectory in the undulator behaves in a complicated way. This note explores these trajectories.

1 Introduction[1]

If the magnet arrays making up the Delta undulator are constructed to have small field rolloff in the direction transverse to the array center line, the Delta can be thought of as two crossed planar variable phase undulators[2]. In this case, the Delta undulator focuses the beam in both the horizontal and vertical directions. The focal length in both directions is the same as for a planar undulator. The focal length is independent of the mode of the undulator and independent of the K value. However, if the field rolloff is significant, the behavior of the Delta is much more complicated. This note calculates the trajectories of the beam in this case.

2 Magnetic Field In The Undulator

2.1 The Scalar Potential

The scalar potential from a single magnet array satisfies Laplace's equation. It varies sinusoidally down the array, it decreases away from the array, and it falls off as one moves away from the center line of the array in the transverse direction. In the coordinate system shown in figure 1, the scalar potential has the form

$$\phi = \phi_0 \cos(k_s s) \exp\left(-k_r r\right) \cos\left(k_u \left(z - z_0\right)\right) \tag{1}$$

where ϕ_0 is a constant, $k_u = 2\pi/\lambda_u$ where λ_u is the undulator period, z is the coordinate down the undulator, and z_0 gives the quadrant position along z. k_r and k_s determine the behavior of the potential in the transverse directions. The potential decreases as one moves radially away from the magnet and this is expressed by $\exp\left(-k_r r\right)$. The potential also decreases as one moves to the side and this is given by the $\cos(k_s s)$ dependence over a limited range of s. Since the Laplacian of the potential is zero, we have the constraint

$$-k_s^2 + k_r^2 - k_u^2 = 0 \tag{2}$$

We use this form of the potential to calculate the magnetic scalar potential in the undulator by rotating the four quadrants and summing their rotated scalar potentials. The Delta undulator is

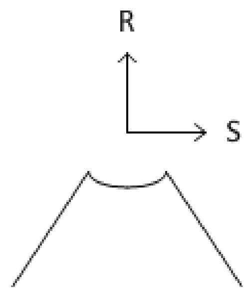

Figure 1: Coordinate system for the scalar potential of a single quadrant.

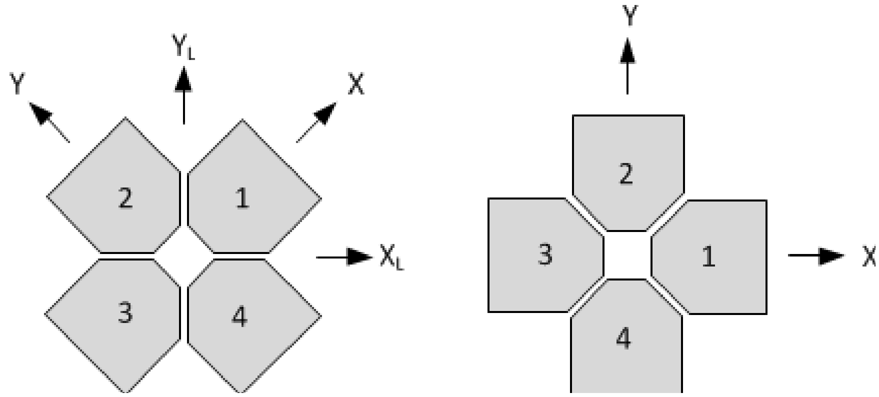

Figure 2: The left side of the figure shows the Delta undulator in its configuration in the tunnel where y_L is up, z is in the beam direction, and x_L makes a right handed system. For our calculations, we use the rotated coordinate system on the right, where x is along the line pointing from quadrant 3 to quadrant 1, y is along the line pointing from quadrant 4 to quadrant 2, and z makes a right handed system.

oriented as shown on the left side of figure 2. In the laboratory, z is along the beam direction, y_L is up, and x_L makes a right handed system. For our calculations, it is more convenient to use the rotated system x, y, z, where x is in the direction from quadrant 3 to quadrant 1, y is in the direction from quadrant 4 to quadrant 2, and z is in the beam direction. Using equation 1, the scalar potential for each of the quadrants in the x, y, z system is

$$\phi_1(x,y,z) = \phi_{0Q} \cos(k_s y) \exp(k_r x) \cos(k_u (z - z_{01})) \quad (3)$$
$$\phi_2(x,y,z) = \phi_{0Q} \cos(k_s x) \exp(k_r y) \cos(k_u (z - z_{02})) \quad (4)$$
$$\phi_3(x,y,z) = -\phi_{0Q} \cos(k_s y) \exp(-k_r x) \cos(k_u (z - z_{03})) \quad (5)$$
$$\phi_4(x,y,z) = -\phi_{0Q} \cos(k_s x) \exp(-k_r y) \cos(k_u (z - z_{04})) \quad (6)$$

where z_{0i} is the longitudinal shift of quadrant i, and ϕ_{0Q} is the amplitude of the scalar potential of all the identical quadrants on the axis of the undulator where $x = 0$ and $y = 0$. Quadrants 3

[1] Work supported in part by the DOE Contract DE-AC02-76SF00515. This work was performed in support of the LCLS project at SLAC.
[2] Z. Wolf, "Beam Trajectories In The Delta Undulator I", LCLS-TN-15-3, March, 2015.

and 4 are loaded with opposite polarity magnets as quadrants 1 and 2 in order to make a vertical field planar undulator in the laboratory frame when all the rows are aligned. This accounts for the minus signs, $-\phi_{0Q}$, in the potentials for quadrants 3 and 4.

The scalar potential for the undulator is the sum of the scalar potentials for the quadrants. Quadrants 1 and 3 both depend on x, and quadrants 2 and 4 both depend on y. We first add the scalar potentials for quadrants 1 and 3, and then add the scalar potentials for quadrants 2 and 4, and then add the sums to get the scalar potential for the whole undulator. We will interpret this as forming the entire undulator from two crossed planar adjustable phase undulators.

The scalar potential for the combination of quadrants 1 and 3 is given by

$$\phi_{13} = \phi_{0Q} \cos(k_s y) \exp(k_r x) \cos(k_u (z - z_{01})) - \phi_{0Q} \cos(k_s y) \exp(-k_r x) \cos(k_u (z - z_{03})) \tag{7}$$

Let

$$z_{01} = Z_{13} + \frac{\Delta_{13}}{2} \tag{8}$$

$$z_{03} = Z_{13} - \frac{\Delta_{13}}{2} \tag{9}$$

where

$$Z_{13} = \frac{z_{01} + z_{03}}{2} \tag{10}$$

is the average z-position of the quadrants, and

$$\Delta_{13} = z_{01} - z_{03} \tag{11}$$

is the z-shift between the quadrants. With these definitions, the scalar potential for the pair of quadrants becomes

$$\begin{aligned}\phi_{13} = {} & 2\phi_{0Q} \cos(k_s y) \sinh(k_r x) \cos\left(k_u \frac{\Delta_{13}}{2}\right) \cos(k_u (z - Z_{13})) \\ & + 2\phi_{0Q} \cos(k_s y) \cosh(k_r x) \sin\left(k_u \frac{\Delta_{13}}{2}\right) \sin(k_u (z - Z_{13}))\end{aligned} \tag{12}$$

This is the scalar potential for a planar adjustable phase undulator[3]. The full range of amplitudes and phases are covered if the range of Δ_{13} includes $\left[-\frac{\lambda_u}{2}, \frac{\lambda_u}{2}\right]$ and the range of Z_{13} includes $\left[-\frac{\lambda_u}{2}, \frac{\lambda_u}{2}\right]$.

Similarly, the scalar potential for the combination of quadrants 2 and 4 is given by

$$\phi_{24} = \phi_{0Q} \cos(k_s x) \exp(k_r y) \cos(k_u (z - z_{02})) - \phi_{0Q} \cos(k_s x) \exp(-k_r y) \cos(k_u (z - z_{04})) \tag{13}$$

Let

$$z_{02} = Z_{24} + \frac{\Delta_{24}}{2} \tag{14}$$

$$z_{04} = Z_{24} - \frac{\Delta_{24}}{2} \tag{15}$$

where

$$Z_{24} = \frac{z_{02} + z_{04}}{2} \tag{16}$$

is the average z-position of the quadrants, and

$$\Delta_{24} = z_{02} - z_{04} \tag{17}$$

[3] Z. Wolf, "Variable Phase PPM Undulator Study", LCLS-TN-11-1, May, 2011.

is the z-shift between the quadrants.

With these definitions, the scalar potential for the pair of quadrants becomes

$$\begin{aligned}\phi_{24} &= 2\phi_{0Q}\cos(k_s x)\sinh(k_r y)\cos\left(k_u\frac{\Delta_{24}}{2}\right)\cos\left(k_u\left(z-Z_{24}\right)\right)\\ &\quad +2\phi_{0Q}\cos(k_s x)\cosh(k_r y)\sin\left(k_u\frac{\Delta_{24}}{2}\right)\sin\left(k_u\left(z-Z_{24}\right)\right)\end{aligned} \tag{18}$$

This is again the potential for a planar adjustable phase undulator.

The scalar potential for the undulator is the sum of the scalar potentials for the quadrant pairs.

$$\phi = \phi_{13} + \phi_{24} \tag{19}$$

Performing the sum, we find

$$\begin{aligned}\phi &= 2\phi_{0Q}\cos(k_s y)\sinh(k_r x)\cos\left(k_u\frac{\Delta_{13}}{2}\right)\cos\left(k_u\left(z-Z_{13}\right)\right)\\ &\quad +2\phi_{0Q}\cos(k_s y)\cosh(k_r x)\sin\left(k_u\frac{\Delta_{13}}{2}\right)\sin\left(k_u\left(z-Z_{13}\right)\right)\\ &\quad +2\phi_{0Q}\cos(k_s x)\sinh(k_r y)\cos\left(k_u\frac{\Delta_{24}}{2}\right)\cos\left(k_u\left(z-Z_{24}\right)\right)\\ &\quad +2\phi_{0Q}\cos(k_s x)\cosh(k_r y)\sin\left(k_u\frac{\Delta_{24}}{2}\right)\sin\left(k_u\left(z-Z_{24}\right)\right)\end{aligned} \tag{20}$$

By putting in the various values for Z_{13}, Z_{24}, Δ_{13}, and Δ_{24}, we get the scalar potential for the various undulator modes at different K values.

2.2 Magnetic Field In The Undulator Coordinate System

The magnetic field in the undulator is given by $B = \nabla\phi$. Taking the gradient we find

$$\begin{aligned}B_x(x,y,z) &= 2\phi_{0Q}k_u[\frac{k_r}{k_u}\cos\left(k_u\frac{\Delta_{13}}{2}\right)\cos(k_s y)\cosh(k_r x)\cos\left(k_u\left(z-Z_{13}\right)\right)\\ &\quad +\frac{k_r}{k_u}\sin\left(k_u\frac{\Delta_{13}}{2}\right)\cos(k_s y)\sinh(k_r x)\sin\left(k_u\left(z-Z_{13}\right)\right)\\ &\quad -\frac{k_s}{k_u}\cos\left(k_u\frac{\Delta_{24}}{2}\right)\sin(k_s x)\sinh(k_r y)\cos\left(k_u\left(z-Z_{24}\right)\right)\\ &\quad -\frac{k_s}{k_u}\sin\left(k_u\frac{\Delta_{24}}{2}\right)\sin(k_s x)\cosh(k_r y)\sin\left(k_u\left(z-Z_{24}\right)\right)]\end{aligned} \tag{21}$$

$$\begin{aligned}B_y(x,y,z) &= 2\phi_{0Q}k_u[-\frac{k_s}{k_u}\cos\left(k_u\frac{\Delta_{13}}{2}\right)\sin(k_s y)\sinh(k_r x)\cos\left(k_u\left(z-Z_{13}\right)\right)\\ &\quad -\frac{k_s}{k_u}\sin\left(k_u\frac{\Delta_{13}}{2}\right)\sin(k_s y)\cosh(k_r x)\sin\left(k_u\left(z-Z_{13}\right)\right)\\ &\quad +\frac{k_r}{k_u}\cos\left(k_u\frac{\Delta_{24}}{2}\right)\cos(k_s x)\cosh(k_r y)\cos\left(k_u\left(z-Z_{24}\right)\right)\\ &\quad +\frac{k_r}{k_u}\sin\left(k_u\frac{\Delta_{24}}{2}\right)\cos(k_s x)\sinh(k_r y)\sin\left(k_u\left(z-Z_{24}\right)\right)]\end{aligned} \tag{22}$$

$$
\begin{aligned}
B_z(x,y,z) \;=\;& 2\phi_{0Q}k_u[-\cos\left(k_u\frac{\Delta_{13}}{2}\right)\cos(k_s y)\sinh(k_r x)\sin(k_u(z-Z_{13})) \\
&+\sin\left(k_u\frac{\Delta_{13}}{2}\right)\cos(k_s y)\cosh(k_r x)\cos(k_u(z-Z_{13})) \\
&-\cos\left(k_u\frac{\Delta_{24}}{2}\right)\cos(k_s x)\sinh(k_r y)\sin(k_u(z-Z_{24})) \\
&+\sin\left(k_u\frac{\Delta_{24}}{2}\right)\cos(k_s x)\cosh(k_r y)\cos(k_u(z-Z_{24}))]
\end{aligned} \tag{23}
$$

In order to simplify these formulas further, let

$$B_0 = 2\phi_{0Q}k_u \tag{24}$$

$$B_{xc} = B_0\cos\left(k_u\frac{\Delta_{13}}{2}\right) \tag{25}$$

$$B_{xs} = B_0\sin\left(k_u\frac{\Delta_{13}}{2}\right) \tag{26}$$

$$B_{yc} = B_0\cos\left(k_u\frac{\Delta_{24}}{2}\right) \tag{27}$$

$$B_{ys} = B_0\sin\left(k_u\frac{\Delta_{24}}{2}\right) \tag{28}$$

With these substitutions, the fields become

$$
\begin{aligned}
B_x(x,y,z) \;=\;& \frac{k_r}{k_u}B_{xc}\cos(k_s y)\cosh(k_r x)\cos(k_u(z-Z_{13})) \\
&+\frac{k_r}{k_u}B_{xs}\cos(k_s y)\sinh(k_r x)\sin(k_u(z-Z_{13})) \\
&-\frac{k_s}{k_u}B_{yc}\sin(k_s x)\sinh(k_r y)\cos(k_u(z-Z_{24})) \\
&-\frac{k_s}{k_u}B_{ys}\sin(k_s x)\cosh(k_r y)\sin(k_u(z-Z_{24}))
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
B_y(x,y,z) \;=\;& -\frac{k_s}{k_u}B_{xc}\sin(k_s y)\sinh(k_r x)\cos(k_u(z-Z_{13})) \\
&-\frac{k_s}{k_u}B_{xs}\sin(k_s y)\cosh(k_r x)\sin(k_u(z-Z_{13})) \\
&+\frac{k_r}{k_u}B_{yc}\cos(k_s x)\cosh(k_r y)\cos(k_u(z-Z_{24})) \\
&+\frac{k_r}{k_u}B_{ys}\cos(k_s x)\sinh(k_r y)\sin(k_u(z-Z_{24}))
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
B_z(x,y,z) \;=\;& -B_{xc}\cos(k_s y)\sinh(k_r x)\sin(k_u(z-Z_{13})) \\
&+B_{xs}\cos(k_s y)\cosh(k_r x)\cos(k_u(z-Z_{13})) \\
&-B_{yc}\cos(k_s x)\sinh(k_r y)\sin(k_u(z-Z_{24})) \\
&+B_{ys}\cos(k_s x)\cosh(k_r y)\cos(k_u(z-Z_{24}))
\end{aligned} \tag{31}
$$

We can change the origin of z such that

$$z = z' + Z_{13} \tag{32}$$

If we let

$$\delta = k_u \left(Z_{13} - Z_{24} \right) \tag{33}$$

and drop the prime, we write the fields as a function of position as

$$\begin{aligned} B_x\left(x,y,z\right) &= B_{xc}\frac{k_r}{k_u}\cos(k_s y)\cosh\left(k_r x\right)\cos\left(k_u z\right) \\ &+B_{xs}\frac{k_r}{k_u}\cos(k_s y)\sinh\left(k_r x\right)\sin\left(k_u z\right) \\ &-B_{yc}\frac{k_s}{k_u}\sin(k_s x)\sinh\left(k_r y\right)\cos\left(k_u z+\delta\right) \\ &-B_{ys}\frac{k_s}{k_u}\sin(k_s x)\cosh\left(k_r y\right)\sin\left(k_u z+\delta\right) \end{aligned} \tag{34}$$

$$\begin{aligned} B_y\left(x,y,z\right) &= -B_{xc}\frac{k_s}{k_u}\sin(k_s y)\sinh\left(k_r x\right)\cos\left(k_u z\right) \\ &-B_{xs}\frac{k_s}{k_u}\sin(k_s y)\cosh\left(k_r x\right)\sin\left(k_u z\right) \\ &+B_{yc}\frac{k_r}{k_u}\cos(k_s x)\cosh\left(k_r y\right)\cos\left(k_u z+\delta\right) \\ &+B_{ys}\frac{k_r}{k_u}\cos(k_s x)\sinh\left(k_r y\right)\sin\left(k_u z+\delta\right) \end{aligned} \tag{35}$$

$$\begin{aligned} B_z\left(x,y,z\right) &= -B_{xc}\cos(k_s y)\sinh\left(k_r x\right)\sin\left(k_u z\right) \\ &+B_{xs}\cos(k_s y)\cosh\left(k_r x\right)\cos\left(k_u z\right) \\ &-B_{yc}\cos(k_s x)\sinh\left(k_r y\right)\sin\left(k_u z+\delta\right) \\ &+B_{ys}\cos(k_s x)\cosh\left(k_r y\right)\cos\left(k_u z+\delta\right) \end{aligned} \tag{36}$$

2.3 Transverse Gradients Of The Field

We will need the partial derivative of the fields in the x and y directions. Calculating these transverse gradients, we find

$$\begin{aligned} \frac{\partial B_x}{\partial x}\left(x,y,z\right) &= B_{xc}\frac{k_r^2}{k_u}\cos(k_s y)\sinh\left(k_r x\right)\cos\left(k_u z\right) \\ &+B_{xs}\frac{k_r^2}{k_u}\cos(k_s y)\cosh\left(k_r x\right)\sin\left(k_u z\right) \\ &-B_{yc}\frac{k_s^2}{k_u}\cos(k_s x)\sinh\left(k_r y\right)\cos\left(k_u z+\delta\right) \\ &-B_{ys}\frac{k_s^2}{k_u}\cos(k_s x)\cosh\left(k_r y\right)\sin\left(k_u z+\delta\right) \end{aligned} \tag{37}$$

$$\begin{aligned} \frac{\partial B_x}{\partial y}\left(x,y,z\right) &= -B_{xc}\frac{k_r k_s}{k_u}\sin(k_s y)\cosh\left(k_r x\right)\cos\left(k_u z\right) \\ &-B_{xs}\frac{k_r k_s}{k_u}\sin(k_s y)\sinh\left(k_r x\right)\sin\left(k_u z\right) \\ &-B_{yc}\frac{k_r k_s}{k_u}\sin(k_s x)\cosh\left(k_r y\right)\cos\left(k_u z+\delta\right) \\ &-B_{ys}\frac{k_r k_s}{k_u}\sin(k_s x)\sinh\left(k_r y\right)\sin\left(k_u z+\delta\right) \end{aligned} \tag{38}$$

$$\frac{\partial B_y}{\partial x}(x,y,z) = -B_{xc}\frac{k_r k_s}{k_u}\sin(k_s y)\cosh(k_r x)\cos(k_u z) -B_{xs}\frac{k_r k_s}{k_u}\sin(k_s y)\sinh(k_r x)\sin(k_u z) -B_{yc}\frac{k_r k_s}{k_u}\sin(k_s x)\cosh(k_r y)\cos(k_u z+\delta) -B_{ys}\frac{k_r k_s}{k_u}\sin(k_s x)\sinh(k_r y)\sin(k_u z+\delta) \quad (39)$$

$$\frac{\partial B_y}{\partial y}(x,y,z) = -B_{xc}\frac{k_s^2}{k_u}\cos(k_s y)\sinh(k_r x)\cos(k_u z) -B_{xs}\frac{k_s^2}{k_u}\cos(k_s y)\cosh(k_r x)\sin(k_u z) +B_{yc}\frac{k_r^2}{k_u}\cos(k_s x)\sinh(k_r y)\cos(k_u z+\delta) +B_{ys}\frac{k_r^2}{k_u}\cos(k_s x)\cosh(k_r y)\sin(k_u z+\delta) \quad (40)$$

2.4 Fields And Transverse Gradients Near The Undulator Axis

We assume we are only operating near the undulator axis where $k_s x \ll 1$, $k_s y \ll 1$, $k_r x \ll 1$, $k_r y \ll 1$. In this case, we expand the functions of these quantities keeping only the first order terms.

$$B_x(x,y,z) = B_{xc}\frac{k_r}{k_u}\cos(k_u z) + B_{xs}\frac{k_r}{k_u}k_r x\sin(k_u z) - B_{ys}\frac{k_s}{k_u}k_s x\sin(k_u z+\delta) \quad (41)$$

$$B_y(x,y,z) = -B_{xs}\frac{k_s}{k_u}k_s y\sin(k_u z) + B_{yc}\frac{k_r}{k_u}\cos(k_u z+\delta) + B_{ys}\frac{k_r}{k_u}k_r y\sin(k_u z+\delta) \quad (42)$$

$$B_z(x,y,z) = -B_{xc}k_r x\sin(k_u z) + B_{xs}\cos(k_u z) - B_{yc}k_r y\sin(k_u z+\delta) + B_{ys}\cos(k_u z+\delta) \quad (43)$$

$$\frac{\partial B_x}{\partial x}(x,y,z) = B_{xc}\frac{k_r^2}{k_u}k_r x\cos(k_u z) + B_{xs}\frac{k_r^2}{k_u}\sin(k_u z) - B_{yc}\frac{k_s^2}{k_u}k_r y\cos(k_u z+\delta) - B_{ys}\frac{k_s^2}{k_u}\sin(k_u z+\delta) \quad (44)$$

$$\frac{\partial B_x}{\partial y}(x,y,z) = -B_{xc}\frac{k_r k_s}{k_u}k_s y\cos(k_u z) - B_{yc}\frac{k_r k_s}{k_u}k_s x\cos(k_u z+\delta) \quad (45)$$

$$\frac{\partial B_y}{\partial x}(x,y,z) = -B_{xc}\frac{k_r k_s}{k_u}k_s y\cos(k_u z) - B_{yc}\frac{k_r k_s}{k_u}k_s x\cos(k_u z+\delta) \quad (46)$$

$$\frac{\partial B_y}{\partial y}(x,y,z) = -B_{xc}\frac{k_s^2}{k_u}k_r x\cos(k_u z) - B_{xs}\frac{k_s^2}{k_u}\sin(k_u z) + B_{yc}\frac{k_r^2}{k_u}k_r y\cos(k_u z+\delta) + B_{ys}\frac{k_r^2}{k_u}\sin(k_u z+\delta) \quad (47)$$

3 Beam Trajectories

3.1 Equations Of Motion

The trajectory of a charged particle beam in the undulator is determined by the Lorentz force law

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{v}\times\mathbf{B}) \quad (48)$$

where $\mathbf{p} = \gamma m \mathbf{v}$, $\gamma = \left(1-\beta^2\right)^{-1}$, and m is the particle rest mass. The energy of the particle is constant in the magnetic field of the undulator, except for radiation losses which we neglect. With constant energy, γ is constant. The Lorentz force law becomes

$$\dot{\mathbf{v}} = \frac{q}{\gamma m}\left(\mathbf{v} \times \mathbf{B}\right) \tag{49}$$

With the substitution $d/dt = v_z d/dz$, the changes in the individual velocity components with z are given by

$$\begin{aligned} \frac{dv_x}{dz} &= \frac{q}{\gamma m v_z}\left(v_y B_z - v_z B_y\right) & (50)\\ \frac{dv_y}{dz} &= \frac{q}{\gamma m v_z}\left(v_z B_x - v_x B_z\right) & (51)\\ \frac{dv_z}{dz} &= \frac{q}{\gamma m v_z}\left(v_x B_y - v_y B_x\right) & (52) \end{aligned}$$

3.2 Iterative Solution

We solve these equations iteratively by expanding in powers of a small parameter. Let

$$\epsilon = \frac{qB_0}{\gamma m k_u c} \tag{53}$$

be the small dimensionless expansion parameter. In terms of this parameter, the equations of motion are

$$\begin{aligned} \frac{dv_x}{dz} &= \epsilon \frac{k_u c}{B_0 v_z}\left(v_y B_z - v_z B_y\right) & (54)\\ \frac{dv_y}{dz} &= \epsilon \frac{k_u c}{B_0 v_z}\left(v_z B_x - v_x B_z\right) & (55)\\ \frac{dv_z}{dz} &= \epsilon \frac{k_u c}{B_0 v_z}\left(v_x B_y - v_y B_x\right) & (56) \end{aligned}$$

At this point we could divide all velocities by c, divide all positions by $1/k_u$, and divide all magnetic fields by B_0 in order to eliminate the factor $\frac{k_u c}{B_0 v_z}$. We think of small quantities in terms of these dimensionless variables. In the interest of clarity, however, we keep the factor and continue to use mks units, so there is no confusion about the meaning of a quantity.

Expanding the particle position in terms of the expansion parameter, we have

$$\begin{aligned} x &= x_{(0)} + \epsilon x_{(1)} + \epsilon^2 x_{(2)} + \cdots & (57)\\ y &= y_{(0)} + \epsilon y_{(1)} + \epsilon^2 y_{(2)} + \cdots & (58)\\ z &= z & (59) \end{aligned}$$

where the last line indicates that the z-position of the particle is the independent variable. The numbers in parenthesis indicate the order of the expansion. The velocities are

$$\begin{aligned} v_x &= v_{x(0)} + \epsilon v_{x(1)} + \epsilon^2 v_{x(2)} + \cdots & (60)\\ v_y &= v_{y(0)} + \epsilon v_{y(1)} + \epsilon^2 v_{y(2)} + \cdots & (61)\\ v_z &= v_{z(0)} + \epsilon v_{z(1)} + \epsilon^2 v_{z(2)} + \cdots & (62) \end{aligned}$$

The magnetic fields are given by

$$\begin{aligned} B_x &= B_x|_0 + \epsilon \partial_x B_x|_0 x_{(1)} + \epsilon \partial_y B_x|_0 y_{(1)} + \cdots & (63)\\ B_y &= B_y|_0 + \epsilon \partial_x B_y|_0 x_{(1)} + \epsilon \partial_y B_y|_0 y_{(1)} + \cdots & (64)\\ B_z &= B_z|_0 + \epsilon \partial_x B_z|_0 x_{(1)} + \epsilon \partial_y B_z|_0 y_{(1)} + \cdots & (65) \end{aligned}$$

where

$$\begin{aligned} B_i|_0 &= B_i(x_{(0)}, y_{(0)}, z) && (66)\\ \partial_j B_i|_0 &= \frac{\partial B_i}{\partial x_j}(x_{(0)}, y_{(0)}, z) && (67)\end{aligned}$$

where $i = x, y, z$, $j = x, y$.

The transverse positions are found from the velocities as follows:

$$\begin{aligned} v_x &= \frac{dx}{dz}\frac{dz}{dt} && (68)\\ x(z) &= x_0 + \int_0^z v_x(z')\frac{1}{v_z(z')}dz' \\ &= x_0 + \int_0^z \left(v_{x(0)} + \epsilon v_{x(1)} + \cdots\right)\frac{1}{v_{z(0)}}\left(1 - \epsilon\frac{v_{z(1)}}{v_{z(0)}} + \cdots\right)dz' \\ &= x_0 + \frac{v_{x(0)}}{v_{z(0)}}z + \epsilon\int_0^z \left(\frac{v_{x(1)}}{v_{z(0)}} - \frac{v_{x(0)}v_{z(1)}}{v_{z(0)}^2}\right)dz' + \cdots && (69)\end{aligned}$$

In these equations x_0 is the constant initial position where the beam enters the undulator. Similarly

$$y(z) = y_0 + \frac{v_{y(0)}}{v_{z(0)}}z + \epsilon\int_0^z \left(\frac{v_{y(1)}}{v_{z(0)}} - \frac{v_{y(0)}v_{z(1)}}{v_{z(0)}^2}\right)dz' + \cdots \tag{70}$$

We see that

$$\begin{aligned} x_{(0)} &= x_0 + \frac{v_{x(0)}}{v_{z(0)}}z && (71)\\ y_{(0)} &= y_0 + \frac{v_{y(0)}}{v_{z(0)}}z && (72)\end{aligned}$$

and

$$\begin{aligned} x_{(1)} &= \int_0^z \left(\frac{v_{x(1)}}{v_{z(0)}} - \frac{v_{x(0)}v_{z(1)}}{v_{z(0)}^2}\right)dz' && (73)\\ y_{(1)} &= \int_0^z \left(\frac{v_{y(1)}}{v_{z(0)}} - \frac{v_{y(0)}v_{z(1)}}{v_{z(0)}^2}\right)dz' && (74)\end{aligned}$$

With the expansions in the small parameter ϵ, the equations of motion become

$$\begin{aligned} &\frac{dv_{x(0)}}{dz} + \epsilon\frac{dv_{x(1)}}{dz} + \epsilon^2\frac{dv_{x(2)}}{dz} + \cdots \\ =\; &\epsilon\frac{k_u c}{B_0 v_{z(0)}}\left[1 - \epsilon\frac{v_{z(1)}}{v_{z(0)}} + \cdots\right] \\ &\times\{(v_{y(0)} + \epsilon v_{y(1)} + \cdots)(B_z|_0 + \epsilon\partial_x B_z|_0 x_{(1)} + \epsilon\partial_y B_z|_0 y_{(1)} + \cdots) \\ &-(v_{z(0)} + \epsilon v_{z(1)} + \cdots)(B_y|_0 + \epsilon\partial_x B_y|_0 x_{(1)} + \epsilon\partial_y B_y|_0 y_{(1)} + \cdots)\} && (75)\end{aligned}$$

$$\begin{aligned} &\frac{dv_{y(0)}}{dz} + \epsilon\frac{dv_{y(1)}}{dz} + \epsilon^2\frac{dv_{y(2)}}{dz} + \cdots \\ =\; &\epsilon\frac{k_u c}{B_0 v_{z(0)}}\left[1 - \epsilon\frac{v_{z(1)}}{v_{z(0)}} + \cdots\right] \\ &\times\{(v_{z(0)} + \epsilon v_{z(1)} + \cdots)(B_x|_0 + \epsilon\partial_x B_x|_0 x_{(1)} + \epsilon\partial_y B_x|_0 y_{(1)} + \cdots) \\ &-(v_{x(0)} + \epsilon v_{x(1)} + \cdots)(B_z|_0 + \epsilon\partial_x B_z|_0 x_{(1)} + \epsilon\partial_y B_z|_0 y_{(1)} + \cdots)\} && (76)\end{aligned}$$

$$
\begin{aligned}
& \frac{dv_{z(0)}}{dz} + \epsilon\frac{dv_{z(1)}}{dz} + \epsilon^2\frac{dv_{z(2)}}{dz} + \cdots \\
= \quad & \epsilon\frac{k_u c}{B_0 v_{z(0)}}\left[1 - \epsilon\frac{v_{z(1)}}{v_{z(0)}} + \cdots\right] \\
& \times\{(v_{x(0)} + \epsilon v_{x(1)} + \cdots)(B_y|_0 + \epsilon\partial_x B_y|_0 x_{(1)} + \epsilon\partial_y B_y|_0 y_{(1)} + \cdots) \\
& -(v_{y(0)} + \epsilon v_{y(1)} + \cdots)(B_x|_0 + \epsilon\partial_x B_x|_0 x_{(1)} + \epsilon\partial_y B_x|_0 y_{(1)} + \cdots)\} \qquad (77)
\end{aligned}
$$

We now proceed to solve these equations for zeroth, first, and second orders.

3.2.1 Zeroth Order

To zeroth order in ϵ, the equations of motion are

$$\frac{dv_{x(0)}}{dz} = 0 \qquad (78)$$

$$\frac{dv_{y(0)}}{dz} = 0 \qquad (79)$$

$$\frac{dv_{z(0)}}{dz} = 0 \qquad (80)$$

The solutions can be written by inspection

$$v_{x(0)} = v_{x0} \qquad (81)$$

$$v_{y(0)} = v_{y0} \qquad (82)$$

$$v_{z(0)} = v_{z0} \qquad (83)$$

where the v_{x0}, v_{y0}, v_{z0} are the constant initial velocities. The corresponding zeroth order transverse positions are

$$x_{(0)} = x_0 + \frac{v_{x0}}{v_{z0}}z \qquad (84)$$

$$y_{(0)} = y_0 + \frac{v_{y0}}{v_{z0}}z \qquad (85)$$

where the x_0 and y_0 have been previously defined to be the initial positions.

At this point, we make a simplifying assumption. We assume that all particles have zero initial transverse velocity. In this case

$$v_{x(0)} = 0 \qquad (86)$$

$$v_{y(0)} = 0 \qquad (87)$$

$$v_{z(0)} = v_{z0} \qquad (88)$$

and

$$x_{(0)} = x_0 \qquad (89)$$

$$y_{(0)} = y_0 \qquad (90)$$

3.2.2 First Order

To first order in ϵ, the equations of motion are

$$\frac{dv_{x(1)}}{dz} = \frac{k_u c}{B_0 v_{z(0)}}\{v_{y(0)} B_z|_0 - v_{z(0)} B_y|_0\} \tag{91}$$

$$\frac{dv_{y(1)}}{dz} = \frac{k_u c}{B_0 v_{z(0)}}\{v_{z(0)} B_x|_0 - v_{x(0)} B_z|_0\} \tag{92}$$

$$\frac{dv_{z(1)}}{dz} = \frac{k_u c}{B_0 v_{z(0)}}\{v_{x(0)} B_y|_0 - v_{y(0)} B_x|_0\} \tag{93}$$

With zero initial transverse velocity, these equations become

$$\frac{dv_{x(1)}}{dz} = -\frac{k_u c}{B_0} B_y|_0 \tag{94}$$

$$\frac{dv_{y(1)}}{dz} = \frac{k_u c}{B_0} B_x|_0 \tag{95}$$

$$\frac{dv_{z(1)}}{dz} = 0 \tag{96}$$

Inserting the expressions for the fields in the undulator coordinate system, we have

$$\frac{dv_{x(1)}}{dz} = -\frac{k_u c}{B_0}[-B_{xs}\frac{k_s}{k_u} k_s y_0 \sin(k_u z) + B_{yc}\frac{k_r}{k_u}\cos(k_u z + \delta) + B_{ys}\frac{k_r}{k_u} k_r y_0 \sin(k_u z + \delta)] \tag{97}$$

$$\frac{dv_{y(1)}}{dz} = \frac{k_u c}{B_0}[B_{xc}\frac{k_r}{k_u}\cos(k_u z) + B_{xs}\frac{k_r}{k_u} k_r x_0 \sin(k_u z) - B_{ys}\frac{k_s}{k_u} k_s x_0 \sin(k_u z + \delta)] \tag{98}$$

$$\frac{dv_{z(1)}}{dz} = 0 \tag{99}$$

We integrate to find the velocity components to first order:

$$v_{x(1)} = -\frac{k_u c}{B_0}[B_{xs}\frac{k_s^2}{k_u^2} y_0 \cos(k_u z) + B_{yc}\frac{k_r}{k_u^2}\sin(k_u z + \delta) - B_{ys}\frac{k_r^2}{k_u^2} y_0 \cos(k_u z + \delta)] \tag{100}$$

$$v_{y(1)} = \frac{k_u c}{B_0}[B_{xc}\frac{k_r}{k_u^2}\sin(k_u z) - B_{xs}\frac{k_r^2}{k_u^2} x_0 \cos(k_u z) + B_{ys}\frac{k_s^2}{k_u^2} x_0 \cos(k_u z + \delta)] \tag{101}$$

$$v_{z(1)} = 0 \tag{102}$$

We use these first order velocities to find the first order corrections to the transverse position of a particle. Using equation 73 with $v_{x(0)} = 0$, we find

$$\begin{aligned} x_{(1)} &= \int_0^z \left(\frac{v_{x(1)}}{v_{z(0)}}\right) dz' \\ &= -\frac{c}{B_0 v_{z(0)}}[B_{xs}\frac{k_s^2}{k_u^2} y_0 \sin(k_u z) - B_{yc}\frac{k_r}{k_u^2}\cos(k_u z + \delta) - B_{ys}\frac{k_r^2}{k_u^2} y_0 \sin(k_u z + \delta)] \end{aligned} \tag{103}$$

Similarly

$$\begin{aligned} y_{(1)} &= \int_0^z \left(\frac{v_{y(1)}}{v_{z(0)}}\right) dz' \\ &= \frac{c}{B_0 v_{z(0)}}[-B_{xc}\frac{k_r}{k_u^2}\cos(k_u z) - B_{xs}\frac{k_r^2}{k_u^2} x_0 \sin(k_u z) + B_{ys}\frac{k_s^2}{k_u^2} x_0 \sin(k_u z + \delta)] \end{aligned} \tag{104}$$

3.2.3 Second Order

To second order in ϵ, the equations of motion are

$$\begin{aligned}\frac{dv_{x(2)}}{dz} &= \frac{k_u c}{B_0 v_{z(0)}}\{v_{y(1)}B_z|_0 + v_{y(0)}[\partial_x B_z|_0 x_{(1)} + \partial_y B_z|_0 y_{(1)}] \\ &\quad -v_{z(1)}B_y|_0 - v_{z(0)}[\partial_x B_y|_0 x_{(1)} + \partial_y B_y|_0 y_{(1)}] \\ &\quad -\frac{v_{z(1)}}{v_{z(0)}}v_{y(0)}B_z|_0 + \frac{v_{z(1)}}{v_{z(0)}}v_{z(0)}B_y|_0\} \end{aligned} \tag{105}$$

$$\begin{aligned}\frac{dv_{y(2)}}{dz} &= \frac{k_u c}{B_0 v_{z(0)}}\{v_{z(1)}B_x|_0 + v_{z(0)}[\partial_x B_x|_0 x_{(1)} + \partial_y B_x|_0 y_{(1)}] \\ &\quad -v_{x(1)}B_z|_0 - v_{x(0)}[\partial_x B_z|_0 x_{(1)} + \partial_y B_z|_0 y_{(1)}] \\ &\quad -\frac{v_{z(1)}}{v_{z(0)}}v_{z(0)}B_x|_0 + \frac{v_{z(1)}}{v_{z(0)}}v_{x(0)}B_z|_0\} \end{aligned} \tag{106}$$

$$\begin{aligned}\frac{dv_{z(2)}}{dz} &= \frac{k_u c}{B_0 v_{z(0)}}\{v_{x(1)}B_y|_0 + v_{x(0)}[\partial_x B_y|_0 x_{(1)} + \partial_y B_y|_0 y_{(1)}] \\ &\quad -v_{y(1)}B_x|_0 - v_{y(0)}[\partial_x B_x|_0 x_{(1)} + \partial_y B_x|_0 y_{(1)}] \\ &\quad -\frac{v_{z(1)}}{v_{z(0)}}v_{x(0)}B_y|_0 + \frac{v_{z(1)}}{v_{z(0)}}v_{y(0)}B_x|_0\} \end{aligned} \tag{107}$$

Inserting the zero value for the zeroth order transverse velocities and the zero value for the first order longitudinal velocity, we have

$$\frac{dv_{x(2)}}{dz} = \frac{k_u c}{B_0 v_{z(0)}}\{v_{y(1)}B_z|_0 - v_{z(0)}[\partial_x B_y|_0 x_{(1)} + \partial_y B_y|_0 y_{(1)}]\} \tag{108}$$

$$\frac{dv_{y(2)}}{dz} = \frac{k_u c}{B_0 v_{z(0)}}\{v_{z(0)}[\partial_x B_x|_0 x_{(1)} + \partial_y B_x|_0 y_{(1)}] - v_{x(1)}B_z|_0\} \tag{109}$$

$$\frac{dv_{z(2)}}{dz} = \frac{k_u c}{B_0 v_{z(0)}}\{v_{x(1)}B_y|_0 - v_{y(1)}B_x|_0\} \tag{110}$$

We must now insert the expressions for the first order velocities, the fields, and the derivatives of the fields.

$$\begin{aligned}\frac{dv_{x(2)}}{dz} &= \frac{k_u c}{B_0 v_{z(0)}}\{\frac{k_u c}{B_0}[B_{xc}\frac{k_r}{k_u^2}\sin(k_u z) - B_{xs}\frac{k_r^2}{k_u^2}x_0\cos(k_u z) + B_{ys}\frac{k_s^2}{k_u^2}x_0\cos(k_u z+\delta)] \\ &\quad \times[-B_{xc}k_r x_0\sin(k_u z) + B_{xs}\cos(k_u z) - B_{yc}k_r y_0\sin(k_u z+\delta) + B_{ys}\cos(k_u z+\delta)] \\ &\quad -v_{z(0)}[-B_{xc}\frac{k_r k_s}{k_u}k_s y_0\cos(k_u z) - B_{yc}\frac{k_r k_s}{k_u}k_s x_0\cos(k_u z+\delta)] \\ &\quad \times\left(-\frac{c}{B_0 v_{z(0)}}\right)[B_{xs}\frac{k_s^2}{k_u^2}y_0\sin(k_u z) - B_{yc}\frac{k_r}{k_u^2}\cos(k_u z+\delta) - B_{ys}\frac{k_r^2}{k_u^2}y_0\sin(k_u z+\delta)] \\ &\quad -v_{z(0)}[-B_{xc}\frac{k_s^2}{k_u}k_r x_0\cos(k_u z) - B_{xs}\frac{k_s^2}{k_u}\sin(k_u z) + B_{yc}\frac{k_r^2}{k_u}k_r y_0\cos(k_u z+\delta) + B_{ys}\frac{k_r^2}{k_u}\sin(k_u z+\delta)] \\ &\quad \times\left(\frac{c}{B_0 v_{z(0)}}\right)[-B_{xc}\frac{k_r}{k_u^2}\cos(k_u z) - B_{xs}\frac{k_r^2}{k_u^2}x_0\sin(k_u z) + B_{ys}\frac{k_s^2}{k_u^2}x_0\sin(k_u z+\delta)]\} \end{aligned} \tag{111}$$

$$
\begin{aligned}
\frac{dv_{y(2)}}{dz} \quad = \quad & \frac{k_u c}{B_0 v_{z(0)}}\{v_{z(0)}[B_{xc}\frac{k_r^2}{k_u}k_r x_0 \cos(k_u z) + B_{xs}\frac{k_r^2}{k_u}\sin(k_u z) - B_{yc}\frac{k_s^2}{k_u}k_r y_0 \cos(k_u z+\delta) - B_{ys}\frac{k_s^2}{k_u}\sin(k_u z+\delta)] \\
& \times\left(-\frac{c}{B_0 v_{z(0)}}\right)[B_{xs}\frac{k_s^2}{k_u^2}y_0 \sin(k_u z) - B_{yc}\frac{k_r}{k_u^2}\cos(k_u z+\delta) - B_{ys}\frac{k_r^2}{k_u^2}y_0 \sin(k_u z+\delta)] \\
& +v_{z(0)}[-B_{xc}\frac{k_r k_s}{k_u}k_s y_0 \cos(k_u z) - B_{yc}\frac{k_r k_s}{k_u}k_s x_0 \cos(k_u z+\delta)] \\
& \times\left(\frac{c}{B_0 v_{z(0)}}\right)[-B_{xc}\frac{k_r}{k_u^2}\cos(k_u z) - B_{xs}\frac{k_r^2}{k_u^2}x_0 \sin(k_u z) + B_{ys}\frac{k_s^2}{k_u^2}x_0 \sin(k_u z+\delta)] \\
& -\left(-\frac{k_u c}{B_0}\right)[B_{xs}\frac{k_s^2}{k_u^2}y_0 \cos(k_u z) + B_{yc}\frac{k_r}{k_u^2}\sin(k_u z+\delta) - B_{ys}\frac{k_r^2}{k_u^2}y_0 \cos(k_u z+\delta)] \\
& \times[-B_{xc}k_r x_0 \sin(k_u z) + B_{xs}\cos(k_u z) - B_{yc}k_r y_0 \sin(k_u z+\delta) + B_{ys}\cos(k_u z+\delta)]\} \qquad (112)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dv_{z(2)}}{dz} \quad = \quad & \frac{k_u c}{B_0 v_{z(0)}}\{\left(-\frac{k_u c}{B_0}\right)[B_{xs}\frac{k_s^2}{k_u^2}y_0 \cos(k_u z) + B_{yc}\frac{k_r}{k_u^2}\sin(k_u z+\delta) - B_{ys}\frac{k_r^2}{k_u^2}y_0 \cos(k_u z+\delta)] \\
& \times[-B_{xs}\frac{k_s}{k_u}k_s y_0 \sin(k_u z) + B_{yc}\frac{k_r}{k_u}\cos(k_u z+\delta) + B_{ys}\frac{k_r}{k_u}k_r y_0 \sin(k_u z+\delta)] \\
& -\left(\frac{k_u c}{B_0}\right)[B_{xc}\frac{k_r}{k_u^2}\sin(k_u z) - B_{xs}\frac{k_r^2}{k_u^2}x_0 \cos(k_u z) + B_{ys}\frac{k_s^2}{k_u^2}x_0 \cos(k_u z+\delta)] \\
& \times[B_{xc}\frac{k_r}{k_u}\cos(k_u z) + B_{xs}\frac{k_r}{k_u}k_r x_0 \sin(k_u z) - B_{ys}\frac{k_s}{k_u}k_s x_0 \sin(k_u z+\delta)]\} \qquad (113)
\end{aligned}
$$

Noting that we keep the beam near the undulator axis where $k_s x_0 \ll 1$, $k_s y_0 \ll 1$, $k_r x_0 \ll 1$, $k_r y_0 \ll 1$, we can expand these products and keep only the terms up to first order in these quantities.

$$
\begin{aligned}
\frac{dv_{x(2)}}{dz} \quad = \quad & \frac{k_u c}{B_0 v_{z(0)}}\{\left(\frac{c}{B_0}\right)[B_{xc}\frac{k_r}{k_u}\sin(k_u z)] \\
& \times[-B_{xc}k_r x_0 \sin(k_u z) + B_{xs}\cos(k_u z) - B_{yc}k_r y_0 \sin(k_u z+\delta) + B_{ys}\cos(k_u z+\delta)] \\
& +\left(\frac{c}{B_0}\right)[-B_{xs}\frac{k_r^2}{k_u}x_0 \cos(k_u z)][B_{xs}\cos(k_u z) + B_{ys}\cos(k_u z+\delta)] \\
& +\left(\frac{c}{B_0}\right)[B_{ys}\frac{k_s^2}{k_u}x_0 \cos(k_u z+\delta)][B_{xs}\cos(k_u z) + B_{ys}\cos(k_u z+\delta)] \\
& +\left(\frac{c}{B_0}\right)[-B_{xc}\frac{k_r k_s}{k_u^2}k_s y_0 \cos(k_u z) - B_{yc}\frac{k_r k_s}{k_u^2}k_s x_0 \cos(k_u z+\delta)][-B_{yc}\frac{k_r}{k_u}\cos(k_u z+\delta)] \\
& -\left(\frac{c}{B_0}\right)[-B_{xc}\frac{k_s^2}{k_u^2}k_r x_0 \cos(k_u z) - B_{xs}\frac{k_s^2}{k_u^2}\sin(k_u z) + B_{yc}\frac{k_r^2}{k_u^2}k_r y_0 \cos(k_u z+\delta) + B_{ys}\frac{k_r^2}{k_u^2}\sin(k_u z+\delta)] \\
& \times[-B_{xc}\frac{k_r}{k_u}\cos(k_u z)] \\
& -\left(\frac{c}{B_0}\right)[-B_{xs}\frac{k_s^2}{k_u^2}\sin(k_u z) + B_{ys}\frac{k_r^2}{k_u^2}\sin(k_u z+\delta)][-B_{xs}\frac{k_r^2}{k_u}x_0 \sin(k_u z)] \\
& -\left(\frac{c}{B_0}\right)[-B_{xs}\frac{k_s^2}{k_u^2}\sin(k_u z) + B_{ys}\frac{k_r^2}{k_u^2}\sin(k_u z+\delta)][B_{ys}\frac{k_s^2}{k_u}x_0 \sin(k_u z+\delta)]\} \qquad (114)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dv_{y(2)}}{dz} \;=\;& \frac{k_u c}{B_0 v_{z(0)}}\{\left(-\frac{c}{B_0}\right)[B_{xs}\frac{k_r^2}{k_u^2}\sin(k_u z) - B_{ys}\frac{k_s^2}{k_u^2}\sin(k_u z+\delta)][B_{xs}\frac{k_s^2}{k_u}y_0\sin(k_u z)] \\
&+\left(-\frac{c}{B_0}\right)[B_{xc}\frac{k_r^2}{k_u^2}k_r x_0\cos(k_u z) + B_{xs}\frac{k_r^2}{k_u^2}\sin(k_u z) - B_{yc}\frac{k_s^2}{k_u^2}k_r y_0\cos(k_u z+\delta) - B_{ys}\frac{k_s^2}{k_u^2}\sin(k_u z+\delta)] \\
&\times[-B_{yc}\frac{k_r}{k_u}\cos(k_u z+\delta)] \\
&+\left(-\frac{c}{B_0}\right)[B_{xs}\frac{k_r^2}{k_u^2}\sin(k_u z) - B_{ys}\frac{k_s^2}{k_u^2}\sin(k_u z+\delta)][-B_{ys}\frac{k_r^2}{k_u}y_0\sin(k_u z+\delta)] \\
&+\left(\frac{c}{B_0}\right)[-B_{xc}\frac{k_r k_s}{k_u^2}k_s y_0\cos(k_u z) - B_{yc}\frac{k_r k_s}{k_u^2}k_s x_0\cos(k_u z+\delta)][-B_{xc}\frac{k_r}{k_u}\cos(k_u z)] \\
&+\left(\frac{c}{B_0}\right)[B_{xs}\frac{k_s^2}{k_u}y_0\cos(k_u z)][B_{xs}\cos(k_u z) + B_{ys}\cos(k_u z+\delta)] \\
&+\left(\frac{c}{B_0}\right)[B_{yc}\frac{k_r}{k_u}\sin(k_u z+\delta)] \\
&\times[-B_{xc}k_r x_0\sin(k_u z) + B_{xs}\cos(k_u z) - B_{yc}k_r y_0\sin(k_u z+\delta) + B_{ys}\cos(k_u z+\delta)] \\
&+\left(\frac{c}{B_0}\right)[-B_{ys}\frac{k_r^2}{k_u}y_0\cos(k_u z+\delta)][B_{xs}\cos(k_u z) + B_{ys}\cos(k_u z+\delta)]\}
\end{aligned} \tag{115}
$$

$$
\begin{aligned}
\frac{dv_{z(2)}}{dz} \;=\;& \frac{k_u c}{B_0 v_{z(0)}}\{\left(-\frac{c}{B_0}\right)[B_{xs}\frac{k_s^2}{k_u}y_0\cos(k_u z) + B_{yc}\frac{k_r}{k_u}\sin(k_u z+\delta) - B_{ys}\frac{k_r^2}{k_u}y_0\cos(k_u z+\delta)] \\
&\times[B_{yc}\frac{k_r}{k_u}\cos(k_u z+\delta)] \\
&+\left(-\frac{c}{B_0}\right)[B_{yc}\frac{k_r}{k_u}\sin(k_u z+\delta)][-B_{xs}\frac{k_s}{k_u}k_s y_0\sin(k_u z) + B_{ys}\frac{k_r}{k_u}k_r y_0\sin(k_u z+\delta)] \\
&-\left(\frac{c}{B_0}\right)[B_{xc}\frac{k_r}{k_u}\sin(k_u z) - B_{xs}\frac{k_r^2}{k_u}x_0\cos(k_u z) + B_{ys}\frac{k_s^2}{k_u}x_0\cos(k_u z+\delta)][B_{xc}\frac{k_r}{k_u}\cos(k_u z)] \\
&-\left(\frac{c}{B_0}\right)[B_{xc}\frac{k_r}{k_u}\sin(k_u z)][B_{xs}\frac{k_r}{k_u}k_r x_0\sin(k_u z) - B_{ys}\frac{k_s}{k_u}k_s x_0\sin(k_u z+\delta)]\}
\end{aligned} \tag{116}
$$

These equations consist of products of terms that vary sinusoidally in z. They result in terms that have no z-dependence and terms that are oscillating as $\cos(2k_u z)$ or $\sin(2k_u z)$. In this note we only consider the terms that do not depend on z. These terms lead to average trajectory deviations from a straight line. We denote the z-independent terms with $\langle\rangle$.

$$\begin{aligned}
\left\langle \frac{dv_{x(2)}}{dz} \right\rangle &= \frac{k_u c^2}{2B_0^2 v_{z(0)}}\{-B_{xc}\frac{k_r}{k_u}B_{xc}k_r x_0 - B_{xc}\frac{k_r}{k_u}B_{yc}k_r y_0\cos(\delta) - B_{xc}\frac{k_r}{k_u}B_{ys}\sin(\delta) \\
&-B_{xs}\frac{k_r^2}{k_u}x_0 B_{xs} - B_{xs}\frac{k_r^2}{k_u}x_0 B_{ys}\cos(\delta) \\
&+B_{ys}\frac{k_s^2}{k_u}x_0 B_{xs}\cos(\delta) + B_{ys}\frac{k_s^2}{k_u}x_0 B_{ys} \\
&+B_{xc}\frac{k_r k_s}{k_u^2}k_s y_0 B_{yc}\frac{k_r}{k_u}\cos(\delta) + B_{yc}\frac{k_r k_s}{k_u^2}k_s x_0 B_{yc}\frac{k_r}{k_u} \\
&-B_{xc}\frac{k_s^2}{k_u^2}k_r x_0 B_{xc}\frac{k_r}{k_u} + B_{yc}\frac{k_r^2}{k_u^2}k_r y_0 B_{xc}\frac{k_r}{k_u}\cos(\delta) + B_{ys}\frac{k_r^2}{k_u^2}B_{xc}\frac{k_r}{k_u}\sin(\delta) \\
&-B_{xs}\frac{k_s^2}{k_u^2}B_{xs}\frac{k_r^2}{k_u}x_0 + B_{ys}\frac{k_r^2}{k_u^2}B_{xs}\frac{k_r^2}{k_u}x_0\cos(\delta) \\
&+B_{xs}\frac{k_s^2}{k_u^2}B_{ys}\frac{k_s^2}{k_u}x_0\cos(\delta) - B_{ys}\frac{k_r^2}{k_u^2}B_{ys}\frac{k_s^2}{k_u}x_0\} \qquad (117)
\end{aligned}$$

$$\begin{aligned}
\left\langle \frac{dv_{y(2)}}{dz} \right\rangle &= \frac{k_u c^2}{2B_0^2 v_{z(0)}}\{-B_{xs}\frac{k_r^2}{k_u^2}B_{xs}\frac{k_s^2}{k_u}y_0 + B_{ys}\frac{k_s^2}{k_u^2}B_{xs}\frac{k_s^2}{k_u}y_0\cos(\delta) \\
&+B_{xc}\frac{k_r^2}{k_u^2}k_r x_0 B_{yc}\frac{k_r}{k_u}\cos(\delta) - B_{xs}\frac{k_r^2}{k_u^2}B_{yc}\frac{k_r}{k_u}\sin(\delta) - B_{yc}\frac{k_s^2}{k_u^2}k_r y_0 B_{yc}\frac{k_r}{k_u} \\
&+B_{xs}\frac{k_r^2}{k_u^2}B_{ys}\frac{k_r^2}{k_u}y_0\cos(\delta) - B_{ys}\frac{k_s^2}{k_u^2}B_{ys}\frac{k_r^2}{k_u}y_0 \\
&+B_{xc}\frac{k_r k_s}{k_u^2}k_s y_0 B_{xc}\frac{k_r}{k_u} + B_{yc}\frac{k_r k_s}{k_u^2}k_s x_0 B_{xc}\frac{k_r}{k_u}\cos(\delta) \\
&+B_{xs}\frac{k_s^2}{k_u}y_0 B_{xs} + B_{xs}\frac{k_s^2}{k_u}y_0 B_{ys}\cos(\delta) \\
&-B_{yc}\frac{k_r}{k_u}B_{xc}k_r x_0\cos(\delta) + B_{yc}\frac{k_r}{k_u}B_{xs}\sin(\delta) - B_{yc}\frac{k_r}{k_u}B_{yc}k_r y_0 \\
&-B_{ys}\frac{k_r^2}{k_u}y_0 B_{xs}\cos(\delta) - B_{ys}\frac{k_r^2}{k_u}y_0 B_{ys}\} \qquad (118)
\end{aligned}$$

$$\begin{aligned}
\left\langle \frac{dv_{z(2)}}{dz} \right\rangle &= \frac{k_u c^2}{2B_0^2 v_{z(0)}}\{-B_{xs}\frac{k_s^2}{k_u}y_0 B_{yc}\frac{k_r}{k_u}\cos(\delta) + B_{ys}\frac{k_r^2}{k_u}y_0 B_{yc}\frac{k_r}{k_u} \\
&+B_{yc}\frac{k_r}{k_u}B_{xs}\frac{k_s}{k_u}k_s y_0\cos(\delta) - B_{yc}\frac{k_r}{k_u}B_{ys}\frac{k_r}{k_u}k_r y_0 \\
&+B_{xs}\frac{k_r^2}{k_u}x_0 B_{xc}\frac{k_r}{k_u} - B_{ys}\frac{k_s^2}{k_u}x_0 B_{xc}\frac{k_r}{k_u}\cos(\delta) \\
&-B_{xc}\frac{k_r}{k_u}B_{xs}\frac{k_r}{k_u}k_r x_0 + B_{xc}\frac{k_r}{k_u}B_{ys}\frac{k_s}{k_u}k_s x_0\cos(\delta)\} \qquad (119)
\end{aligned}$$

Grouping terms, these equations become

$$\begin{aligned}\left\langle \frac{dv_{x(2)}}{dz} \right\rangle &= \frac{k_u c^2}{2B_0^2 v_{z(0)}}\{-B_{xc}\frac{k_r}{k_u}B_{ys}\sin(\delta) + B_{ys}\frac{k_r^2}{k_u^2}B_{xc}\frac{k_r}{k_u}\sin(\delta) \\ &+k_u x_0[-B_{xc}\frac{k_r^2}{k_u^2}B_{xc} - B_{xs}\frac{k_r^2}{k_u^2}B_{xs} - B_{xs}\frac{k_r^2}{k_u^2}B_{ys}\cos(\delta) + B_{ys}\frac{k_s^2}{k_u^2}B_{xs}\cos(\delta) + B_{ys}\frac{k_s^2}{k_u^2}B_{ys} \\ &+B_{yc}\frac{k_r^2k_s^2}{k_u^4}B_{yc} - B_{xc}\frac{k_r^2k_s^2}{k_u^4}B_{xc} - B_{xs}\frac{k_r^2k_s^2}{k_u^4}B_{xs} + B_{ys}\frac{k_r^4}{k_u^4}B_{xs}\cos(\delta) + B_{xs}\frac{k_s^4}{k_u^4}B_{ys}\cos(\delta) - B_{ys}\frac{k_r^2k_s^2}{k_u^4}B_{ys}] \\ &+k_u y_0[-B_{xc}\frac{k_r^2}{k_u^2}B_{yc}\cos(\delta) + B_{xc}\frac{k_r^2k_s^2}{k_u^4}B_{yc}\cos(\delta) + B_{yc}\frac{k_r^4}{k_u^4}B_{xc}\cos(\delta)]\} \qquad (120)\end{aligned}$$

$$\begin{aligned}\left\langle \frac{dv_{y(2)}}{dz} \right\rangle &= \frac{k_u c^2}{2B_0^2 v_{z(0)}}\{-B_{xs}\frac{k_r^2}{k_u^2}B_{yc}\frac{k_r}{k_u}\sin(\delta) + B_{yc}\frac{k_r}{k_u}B_{xs}\sin(\delta) \\ &+k_u y_0[-B_{xs}\frac{k_r^2k_s^2}{k_u^4}B_{xs} + B_{ys}\frac{k_s^4}{k_u^4}B_{xs}\cos(\delta) - B_{yc}\frac{k_s^2k_r^2}{k_u^4}B_{yc} + B_{xs}\frac{k_r^4}{k_u^4}B_{ys}\cos(\delta) - B_{ys}\frac{k_r^2k_s^2}{k_u^4}B_{ys} \\ &+B_{xc}\frac{k_r^2k_s^2}{k_u^4}B_{xc} + B_{xs}\frac{k_s^2}{k_u^2}B_{ys}\cos(\delta) + B_{xs}\frac{k_s^2}{k_u^2}B_{xs} - B_{yc}\frac{k_r^2}{k_u^2}B_{yc} - B_{ys}\frac{k_r^2}{k_u^2}B_{xs}\cos(\delta) - B_{ys}\frac{k_r^2}{k_u^2}B_{ys}] \\ &+k_u x_0[B_{xc}\frac{k_r^4}{k_u^4}B_{yc}\cos(\delta) + B_{yc}\frac{k_r^2k_s^2}{k_u^4}B_{xc}\cos(\delta) - B_{yc}\frac{k_r^2}{k_u^2}B_{xc}\cos(\delta)]\} \qquad (121)\end{aligned}$$

$$\begin{aligned}\left\langle \frac{dv_{z(2)}}{dz} \right\rangle &= \frac{k_u c^2}{2B_0^2 v_{z(0)}}\{k_u x_0[B_{xs}\frac{k_r^3}{k_u^3}B_{xc} - B_{xc}\frac{k_r^3}{k_u^3}B_{xs} - B_{ys}\frac{k_rk_s^2}{k_u^3}B_{xc}\cos(\delta) + B_{xc}\frac{k_rk_s^2}{k_u^3}B_{ys}\cos(\delta)] \\ &+k_u y_0[-B_{xs}\frac{k_rk_s^2}{k_u^3}B_{yc}\cos(\delta) + B_{yc}\frac{k_rk_s^2}{k_u^3}B_{xs}\cos(\delta) + B_{ys}\frac{k_r^3}{k_u^3}B_{yc} - B_{yc}\frac{k_r^3}{k_u^3}B_{ys}]\} \qquad (122)\end{aligned}$$

Grouping terms further, we have

$$\begin{aligned}\left\langle \frac{dv_{x(2)}}{dz} \right\rangle &= \frac{k_u c^2}{2B_0^2 v_{z(0)}}\{-B_{xc}B_{ys}\frac{k_r}{k_u}\sin(\delta)(1-\frac{k_r^2}{k_u^2}) \\ &+k_u x_0[-B_{xc}^2\frac{k_r^2}{k_u^2}(1+\frac{k_s^2}{k_u^2}) - B_{xs}^2\frac{k_r^2}{k_u^2}(1+\frac{k_s^2}{k_u^2}) + B_{yc}^2\frac{k_r^2k_s^2}{k_u^4} + B_{ys}^2\frac{k_s^2}{k_u^2}(1-\frac{k_r^2}{k_u^2}) \\ &+B_{xs}B_{ys}\cos(\delta)(-\frac{k_r^2}{k_u^2}+\frac{k_s^2}{k_u^2}+\frac{k_r^4}{k_u^4}+\frac{k_s^4}{k_u^4})] \\ &+k_u y_0[B_{xc}B_{yc}\cos(\delta)(-\frac{k_r^2}{k_u^2}+\frac{k_r^2k_s^2}{k_u^4}+\frac{k_r^4}{k_u^4})]\} \qquad (123)\end{aligned}$$

$$\begin{aligned}\left\langle \frac{dv_{y(2)}}{dz} \right\rangle &= \frac{k_u c^2}{2B_0^2 v_{z(0)}}\{B_{yc}B_{xs}\frac{k_r}{k_u}\sin(\delta)(1-\frac{k_r^2}{k_u^2}) \\ &+k_u y_0[-B_{yc}^2\frac{k_r^2}{k_u^2}(1+\frac{k_s^2}{k_u^2}) - B_{ys}^2\frac{k_r^2}{k_u^2}(1+\frac{k_s^2}{k_u^2}) + B_{xc}^2\frac{k_r^2k_s^2}{k_u^4} + B_{xs}^2\frac{k_s^2}{k_u^2}(1-\frac{k_r^2}{k_u^2}) \\ &+B_{ys}B_{xs}\cos(\delta)(-\frac{k_r^2}{k_u^2}+\frac{k_s^2}{k_u^2}+\frac{k_r^4}{k_u^4}+\frac{k_s^4}{k_u^4})] \\ &+k_u x_0[B_{yc}B_{xc}\cos(\delta)(-\frac{k_r^2}{k_u^2}+\frac{k_r^2k_s^2}{k_u^4}+\frac{k_r^4}{k_u^4})]\} \qquad (124)\end{aligned}$$

$$\left\langle \frac{dv_{z(2)}}{dz} \right\rangle = 0 \tag{125}$$

The constraint equation 2 gives

$$-\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2} - 1 = 0 \tag{126}$$

So

$$1 - \frac{k_r^2}{k_u^2} = -\frac{k_s^2}{k_u^2} \tag{127}$$

$$1 + \frac{k_s^2}{k_u^2} = \frac{k_r^2}{k_u^2} \tag{128}$$

$$\begin{aligned} -\frac{k_r^2}{k_u^2} + \frac{k_s^2}{k_u^2} + \frac{k_r^4}{k_u^4} + \frac{k_s^4}{k_u^4} &= -\frac{k_r^2}{k_u^2}(1 - \frac{k_r^2}{k_u^2}) + \frac{k_s^2}{k_u^2}(1 + \frac{k_s^2}{k_u^2}) \\ &= \frac{k_r^2}{k_u^2}\frac{k_s^2}{k_u^2} + \frac{k_s^2}{k_u^2}\frac{k_r^2}{k_u^2} \\ &= 2\frac{k_r^2 k_s^2}{k_u^4} \end{aligned} \tag{129}$$

$$\begin{aligned} -\frac{k_r^2}{k_u^2} + \frac{k_r^2 k_s^2}{k_u^4} + \frac{k_r^4}{k_u^4} &= -\frac{k_r^2}{k_u^2}(1 - \frac{k_r^2}{k_u^2}) + \frac{k_r^2 k_s^2}{k_u^4} \\ &= \frac{k_r^2}{k_u^2}\frac{k_s^2}{k_u^2} + \frac{k_r^2 k_s^2}{k_u^4} \\ &= 2\frac{k_r^2 k_s^2}{k_u^4} \end{aligned} \tag{130}$$

Inserting these relations from the constrain equation gives

$$\begin{aligned} \left\langle \frac{dv_{x(2)}}{dz} \right\rangle &= \frac{k_u c^2}{2B_0^2 v_{z(0)}}\{B_{xc}B_{ys}\frac{k_r k_s^2}{k_u^3}\sin(\delta) \\ &+k_u x_0[-B_{xc}^2\frac{k_r^2}{k_u^2}\frac{k_r^2}{k_u^2} - B_{xs}^2\frac{k_r^2}{k_u^2}\frac{k_r^2}{k_u^2} + B_{yc}^2\frac{k_r^2 k_s^2}{k_u^4} - B_{ys}^2\frac{k_s^2}{k_u^2}\frac{k_s^2}{k_u^2} \\ &+B_{xs}B_{ys}\cos(\delta)2\frac{k_r^2 k_s^2}{k_u^4}] \\ &+k_u y_0[B_{xc}B_{yc}\cos(\delta)2\frac{k_r^2 k_s^2}{k_u^4}]\} \end{aligned} \tag{131}$$

$$\begin{aligned} \left\langle \frac{dv_{y(2)}}{dz} \right\rangle &= \frac{k_u c^2}{2B_0^2 v_{z(0)}}\{-B_{yc}B_{xs}\frac{k_r k_s^2}{k_u^3}\sin(\delta) \\ &+k_u y_0[-B_{yc}^2\frac{k_r^2}{k_u^2}\frac{k_r^2}{k_u^2} - B_{ys}^2\frac{k_r^2}{k_u^2}\frac{k_r^2}{k_u^2} + B_{xc}^2\frac{k_r^2 k_s^2}{k_u^4} - B_{xs}^2\frac{k_s^2}{k_u^2}\frac{k_s^2}{k_u^2} \\ &+B_{ys}B_{xs}\cos(\delta)2\frac{k_r^2 k_s^2}{k_u^4}] \\ &+k_u x_0[B_{yc}B_{xc}\cos(\delta)2\frac{k_r^2 k_s^2}{k_u^4}]\} \end{aligned} \tag{132}$$

$$\left\langle \frac{dv_{z(2)}}{dz} \right\rangle = 0 \tag{133}$$

Using the relations

$$B_{xc}^2 + B_{xs}^2 = B_0^2 \tag{134}$$

$$B_{yc}^2 + B_{ys}^2 = B_0^2 \tag{135}$$

we have

$$\begin{aligned}\left\langle \frac{dv_{x(2)}}{dz} \right\rangle &= \frac{k_u c^2}{2B_0^2 v_{z(0)}}\{B_{xc}B_{ys}\frac{k_r k_s^2}{k_u^3}\sin(\delta) \\ &+k_u x_0[-B_0^2\frac{k_r^4}{k_u^4} + B_{yc}^2\frac{k_r^2 k_s^2}{k_u^4} - (B_0^2 - B_{yc}^2)\frac{k_s^2}{k_u^2}\frac{k_s^2}{k_u^2} \\ &+2B_{xs}B_{ys}\frac{k_r^2 k_s^2}{k_u^4}\cos(\delta)] \\ &+k_u y_0[2B_{xc}B_{yc}\frac{k_r^2 k_s^2}{k_u^4}\cos(\delta)]\}\end{aligned} \tag{136}$$

$$\begin{aligned}\left\langle \frac{dv_{y(2)}}{dz} \right\rangle &= \frac{k_u c^2}{2B_0^2 v_{z(0)}}\{-B_{yc}B_{xs}\frac{k_r k_s^2}{k_u^3}\sin(\delta) \\ &+k_u y_0[-B_0^2\frac{k_r^4}{k_u^4} + B_{xc}^2\frac{k_r^2 k_s^2}{k_u^4} - (B_0^2 - B_{xc}^2)\frac{k_s^2}{k_u^2}\frac{k_s^2}{k_u^2} \\ &+2B_{ys}B_{xs}\frac{k_r^2 k_s^2}{k_u^4}\cos(\delta)] \\ &+k_u x_0[2B_{yc}B_{xc}\frac{k_r^2 k_s^2}{k_u^4}\cos(\delta)]\}\end{aligned} \tag{137}$$

$$\left\langle \frac{dv_{z(2)}}{dz} \right\rangle = 0 \tag{138}$$

Simplifying, we have

$$\begin{aligned}\left\langle \frac{dv_{x(2)}}{dz} \right\rangle &= \frac{k_u c^2}{2B_0^2 v_{z(0)}}\{B_{xc}B_{ys}\frac{k_r k_s^2}{k_u^3}\sin(\delta) \\ &+k_u x_0[-B_0^2\frac{k_r^4}{k_u^4} - B_0^2\frac{k_s^4}{k_u^4} + B_{yc}^2\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2}) + 2B_{xs}B_{ys}\frac{k_r^2 k_s^2}{k_u^4}\cos(\delta)] \\ &+k_u y_0[2B_{xc}B_{yc}\frac{k_r^2 k_s^2}{k_u^4}\cos(\delta)]\}\end{aligned} \tag{139}$$

$$\begin{aligned}\left\langle \frac{dv_{y(2)}}{dz} \right\rangle &= \frac{k_u c^2}{2B_0^2 v_{z(0)}}\{-B_{yc}B_{xs}\frac{k_r k_s^2}{k_u^3}\sin(\delta) \\ &+k_u y_0[-B_0^2\frac{k_r^4}{k_u^4} - B_0^2\frac{k_s^4}{k_u^4} + B_{xc}^2\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2}) + 2B_{ys}B_{xs}\frac{k_r^2 k_s^2}{k_u^4}\cos(\delta)] \\ &+k_u x_0[2B_{yc}B_{xc}\frac{k_r^2 k_s^2}{k_u^4}\cos(\delta)]\}\end{aligned} \tag{140}$$

$$\left\langle \frac{dv_{z(2)}}{dz} \right\rangle = 0 \tag{141}$$

Note that if $k_s = 0$ and $k_r = k_u$, we have

$$\left\langle \frac{dv_{x(2)}}{dz} \right\rangle = -\frac{k_u^2 c^2}{2v_{z(0)}} x_0 \tag{142}$$

$$\left\langle \frac{dv_{y(2)}}{dz} \right\rangle = -\frac{k_u^2 c^2}{2v_{z(0)}} y_0 \tag{143}$$

$$\left\langle \frac{dv_{z(2)}}{dz} \right\rangle = 0 \tag{144}$$

which agrees with a previous result[4].

The second order velocities through the undulator are

$$v_{x(2)} = \left\langle \frac{dv_{x(2)}}{dz} \right\rangle z + osc(2k_u z) \tag{145}$$

$$v_{y(2)} = \left\langle \frac{dv_{y(2)}}{dz} \right\rangle z + osc(2k_u z) \tag{146}$$

$$v_{z(2)} = osc(2k_u z) \tag{147}$$

where $osc(2k_u z)$ represent terms that go as $\cos(2k_u z)$ or $\sin(2k_u z)$.

3.3 Solution To The Equations Of Motion

The velocity of a particle in the undulator can now be given to second order. Using

$$v_x = v_{x(0)} + \epsilon v_{x(1)} + \epsilon^2 v_{x(2)} + \cdots \tag{148}$$

$$v_y = v_{y(0)} + \epsilon v_{y(1)} + \epsilon^2 v_{y(2)} + \cdots \tag{149}$$

$$v_z = v_{z(0)} + \epsilon v_{z(1)} + \epsilon^2 v_{z(2)} + \cdots \tag{150}$$

with

$$\epsilon = \frac{qB_0}{\gamma m k_u c} \tag{151}$$

we get

$$\begin{aligned} v_x &= \frac{qB_0}{\gamma m k_u c}\left[-\frac{k_u c}{B_0}[B_{xs}\frac{k_s^2}{k_u^2} y_0 \cos(k_u z) + B_{yc}\frac{k_r}{k_u^2}\sin(k_u z + \delta) - B_{ys}\frac{k_r^2}{k_u^2} y_0 \cos(k_u z + \delta)]\right] \\ &+ \left(\frac{qB_0}{\gamma m k_u c}\right)^2 \left\langle \frac{dv_{x(2)}}{dz} \right\rangle z + \epsilon^2 osc(2k_u z) + \cdots \end{aligned} \tag{152}$$

$$\begin{aligned} v_y &= \frac{qB_0}{\gamma m k_u c}\left[\frac{k_u c}{B_0}[B_{xc}\frac{k_r}{k_u^2}\sin(k_u z) - B_{xs}\frac{k_r^2}{k_u^2} x_0 \cos(k_u z) + B_{ys}\frac{k_s^2}{k_u^2} x_0 \cos(k_u z + \delta)]\right] \\ &+ \left(\frac{qB_0}{\gamma m k_u c}\right)^2 \left\langle \frac{dv_{y(2)}}{dz} \right\rangle z + \epsilon^2 osc(2k_u z) + \cdots \end{aligned} \tag{153}$$

$$v_z = v_{z0} + \epsilon^2 osc(2k_u z) + \cdots \tag{154}$$

The average velocity of the particle in the undulator is

$$\langle v_x \rangle = \left(\frac{qB_0}{\gamma m k_u c}\right)^2 \left\langle \frac{dv_{x(2)}}{dz} \right\rangle z \tag{155}$$

$$\langle v_y \rangle = \left(\frac{qB_0}{\gamma m k_u c}\right)^2 \left\langle \frac{dv_{y(2)}}{dz} \right\rangle z \tag{156}$$

$$\langle v_z \rangle = v_{z0} \tag{157}$$

[4] Z. Wolf, "Beam Trajectories In The Delta Undulator I", LCLS-TN-15-3, March, 2015.

where

$$\left\langle \frac{dv_{x(2)}}{dz} \right\rangle = \frac{k_u c^2}{2B_0^2 v_{z(0)}}\{B_{xc}B_{ys}\frac{k_r k_s^2}{k_u^3}\sin(\delta)$$
$$+k_u x_0[-B_0^2\frac{k_r^4}{k_u^4} - B_0^2\frac{k_s^4}{k_u^4} + B_{yc}^2\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2}) + 2B_{xs}B_{ys}\frac{k_r^2 k_s^2}{k_u^4}\cos(\delta)]$$
$$+k_u y_0[2B_{xc}B_{yc}\frac{k_r^2 k_s^2}{k_u^4}\cos(\delta)]\} \qquad (158)$$

$$\left\langle \frac{dv_{y(2)}}{dz} \right\rangle = \frac{k_u c^2}{2B_0^2 v_{z(0)}}\{-B_{yc}B_{xs}\frac{k_r k_s^2}{k_u^3}\sin(\delta)$$
$$+k_u y_0[-B_0^2\frac{k_r^4}{k_u^4} - B_0^2\frac{k_s^4}{k_u^4} + B_{xc}^2\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2}) + 2B_{ys}B_{xs}\frac{k_r^2 k_s^2}{k_u^4}\cos(\delta)]$$
$$+k_u x_0[2B_{yc}B_{xc}\frac{k_r^2 k_s^2}{k_u^4}\cos(\delta)]\} \qquad (159)$$

4 Laboratory Coordinate System

4.1 Average Velocity

The undulator frame is rotated by 45° in the laboratory frame. The transformation to the laboratory frame coordinates is

$$x_L = \frac{1}{\sqrt{2}}(x - y) \qquad (160)$$
$$y_L = \frac{1}{\sqrt{2}}(x + y) \qquad (161)$$

and the inverse transformation is

$$x = \frac{1}{\sqrt{2}}(x_L + y_L) \qquad (162)$$
$$y = \frac{1}{\sqrt{2}}(-x_L + y_L) \qquad (163)$$

The transformation of the average velocity is

$$\langle v_{xL} \rangle = \frac{1}{\sqrt{2}}(\langle v_x \rangle - \langle v_y \rangle) \qquad (164)$$
$$\langle v_{yL} \rangle = \frac{1}{\sqrt{2}}(\langle v_x \rangle + \langle v_y \rangle) \qquad (165)$$

Using equations 155 and 156, the average velocities in the laboratory frame are

$$\langle v_{xL} \rangle = \frac{1}{\sqrt{2}}\left(\frac{qB_0}{\gamma m k_u c}\right)^2\left[\left\langle \frac{dv_{x(2)}}{dz} \right\rangle - \left\langle \frac{dv_{y(2)}}{dz} \right\rangle\right] z \qquad (166)$$
$$\langle v_{yL} \rangle = \frac{1}{\sqrt{2}}\left(\frac{qB_0}{\gamma m k_u c}\right)^2\left[\left\langle \frac{dv_{x(2)}}{dz} \right\rangle + \left\langle \frac{dv_{y(2)}}{dz} \right\rangle\right] z \qquad (167)$$

Using the equations for the average derivatives of the velocities, we find

$$\begin{aligned}
\left\langle\frac{dv_{x(2)}}{dz}\right\rangle-\left\langle\frac{dv_{y(2)}}{dz}\right\rangle &= \frac{k_u c^2}{2B_0^2 v_{z(0)}}\{B_{xc}B_{ys}\frac{k_r k_s^2}{k_u^3}\sin(\delta)+B_{yc}B_{xs}\frac{k_r k_s^2}{k_u^3}\sin(\delta)\\
&+k_u\frac{1}{\sqrt{2}}\left(x_{0L}+y_{0L}\right)[-B_0^2\frac{k_r^4}{k_u^4}-B_0^2\frac{k_s^4}{k_u^4}+B_{yc}^2\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2}+\frac{k_r^2}{k_u^2})+2B_{xs}B_{ys}\frac{k_r^2k_s^2}{k_u^4}\cos(\delta)]\\
&-k_u\frac{1}{\sqrt{2}}\left(-x_{0L}+y_{0L}\right)[-B_0^2\frac{k_r^4}{k_u^4}-B_0^2\frac{k_s^4}{k_u^4}+B_{xc}^2\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2}+\frac{k_r^2}{k_u^2})+2B_{ys}B_{xs}\frac{k_r^2k_s^2}{k_u^4}\cos(\delta)]\\
&+k_u\frac{1}{\sqrt{2}}\left(-x_{0L}+y_{0L}\right)[2B_{xc}B_{yc}\frac{k_r^2k_s^2}{k_u^4}\cos(\delta)]\\
&-k_u\frac{1}{\sqrt{2}}\left(x_{0L}+y_{0L}\right)[2B_{yc}B_{xc}\frac{k_r^2k_s^2}{k_u^4}\cos(\delta)]\}
\end{aligned}\tag{168}$$

$$\begin{aligned}
\left\langle\frac{dv_{x(2)}}{dz}\right\rangle+\left\langle\frac{dv_{y(2)}}{dz}\right\rangle &= \frac{k_u c^2}{2B_0^2 v_{z(0)}}\{B_{xc}B_{ys}\frac{k_r k_s^2}{k_u^3}\sin(\delta)-B_{yc}B_{xs}\frac{k_r k_s^2}{k_u^3}\sin(\delta)\\
&+k_u\frac{1}{\sqrt{2}}\left(x_{0L}+y_{0L}\right)[-B_0^2\frac{k_r^4}{k_u^4}-B_0^2\frac{k_s^4}{k_u^4}+B_{yc}^2\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2}+\frac{k_r^2}{k_u^2})+2B_{xs}B_{ys}\frac{k_r^2k_s^2}{k_u^4}\cos(\delta)]\\
&+k_u\frac{1}{\sqrt{2}}\left(-x_{0L}+y_{0L}\right)[-B_0^2\frac{k_r^4}{k_u^4}-B_0^2\frac{k_s^4}{k_u^4}+B_{xc}^2\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2}+\frac{k_r^2}{k_u^2})+2B_{ys}B_{xs}\frac{k_r^2k_s^2}{k_u^4}\cos(\delta)]\\
&+k_u\frac{1}{\sqrt{2}}\left(-x_{0L}+y_{0L}\right)[2B_{xc}B_{yc}\frac{k_r^2k_s^2}{k_u^4}\cos(\delta)]\\
&+k_u\frac{1}{\sqrt{2}}\left(x_{0L}+y_{0L}\right)[2B_{yc}B_{xc}\frac{k_r^2k_s^2}{k_u^4}\cos(\delta)]\}
\end{aligned}\tag{169}$$

Simplifying, we have

$$\begin{aligned}
\left\langle\frac{dv_{x(2)}}{dz}\right\rangle-\left\langle\frac{dv_{y(2)}}{dz}\right\rangle &= \frac{k_u c^2}{2B_0^2 v_{z(0)}}\{(B_{xc}B_{ys}+B_{yc}B_{xs})\frac{k_r k_s^2}{k_u^3}\sin(\delta)\\
&+k_u\sqrt{2}x_{0L}[-B_0^2\frac{k_r^4}{k_u^4}-B_0^2\frac{k_s^4}{k_u^4}+2B_{xs}B_{ys}\frac{k_r^2k_s^2}{k_u^4}\cos(\delta)]\\
&+k_u\frac{1}{\sqrt{2}}x_{0L}\left(B_{yc}^2+B_{xc}^2\right)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2}+\frac{k_r^2}{k_u^2})\\
&+k_u\frac{1}{\sqrt{2}}y_{0L}\left(B_{yc}^2-B_{xc}^2\right)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2}+\frac{k_r^2}{k_u^2})\\
&-k_u\sqrt{2}x_{0L}[2B_{xc}B_{yc}\frac{k_r^2k_s^2}{k_u^4}\cos(\delta)]\}
\end{aligned}\tag{170}$$

$$\begin{aligned}
\left\langle\frac{dv_{x(2)}}{dz}\right\rangle+\left\langle\frac{dv_{y(2)}}{dz}\right\rangle &= \frac{k_u c^2}{2B_0^2 v_{z(0)}}\{(B_{xc}B_{ys}-B_{yc}B_{xs})\frac{k_r k_s^2}{k_u^3}\sin(\delta)\\
&+k_u\sqrt{2}y_{0L}[-B_0^2\frac{k_r^4}{k_u^4}-B_0^2\frac{k_s^4}{k_u^4}+2B_{xs}B_{ys}\frac{k_r^2k_s^2}{k_u^4}\cos(\delta)]\\
&+k_u\frac{1}{\sqrt{2}}x_{0L}\left(B_{yc}^2-B_{xc}^2\right)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2}+\frac{k_r^2}{k_u^2})\\
&+k_u\frac{1}{\sqrt{2}}y_{0L}\left(B_{yc}^2+B_{xc}^2\right)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2}+\frac{k_r^2}{k_u^2})\\
&+k_u\sqrt{2}y_{0L}[2B_{xc}B_{yc}\frac{k_r^2k_s^2}{k_u^4}\cos(\delta)]\}
\end{aligned}\tag{171}$$

Collecting terms, we find

$$\left\langle \frac{dv_{x(2)}}{dz} \right\rangle - \left\langle \frac{dv_{y(2)}}{dz} \right\rangle = \frac{k_u c^2}{2B_0^2 v_{z(0)}} \{ (B_{xc}B_{ys} + B_{yc}B_{xs}) \frac{k_r k_s^2}{k_u^3} \sin(\delta)$$
$$+ k_u \sqrt{2} x_{0L} [-B_0^2 \frac{k_r^4}{k_u^4} - B_0^2 \frac{k_s^4}{k_u^4} + 2 (B_{xs}B_{ys} - B_{xc}B_{yc}) \frac{k_r^2 k_s^2}{k_u^4} \cos(\delta)$$
$$+ \frac{1}{2} \left(B_{yc}^2 + B_{xc}^2 \right) \frac{k_s^2}{k_u^2} (\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})]$$
$$+ k_u \sqrt{2} y_{0L} [\frac{1}{2} \left(B_{yc}^2 - B_{xc}^2 \right) \frac{k_s^2}{k_u^2} (\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})] \} \tag{172}$$

$$\left\langle \frac{dv_{x(2)}}{dz} \right\rangle + \left\langle \frac{dv_{y(2)}}{dz} \right\rangle = \frac{k_u c^2}{2B_0^2 v_{z(0)}} \{ (B_{xc}B_{ys} - B_{yc}B_{xs}) \frac{k_r k_s^2}{k_u^3} \sin(\delta)$$
$$+ k_u \sqrt{2} y_{0L} [-B_0^2 \frac{k_r^4}{k_u^4} - B_0^2 \frac{k_s^4}{k_u^4} + 2 (B_{xs}B_{ys} + B_{xc}B_{yc}) \frac{k_r^2 k_s^2}{k_u^4} \cos(\delta)$$
$$+ \frac{1}{2} \left(B_{yc}^2 + B_{xc}^2 \right) \frac{k_s^2}{k_u^2} (\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})]$$
$$+ k_u \frac{1}{\sqrt{2}} x_{0L} \left(B_{yc}^2 - B_{xc}^2 \right) \frac{k_s^2}{k_u^2} (\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2}) \} \tag{173}$$

Inserting these expressions in the equations for the average velocities gives

$$\langle v_{xL} \rangle = \frac{1}{2\sqrt{2}} \frac{q^2}{\gamma^2 m^2 k_u v_{z(0)}} z$$
$$\times \{ (B_{xc}B_{ys} + B_{yc}B_{xs}) \frac{k_r k_s^2}{k_u^3} \sin(\delta)$$
$$+ k_u \sqrt{2} x_{0L} [-B_0^2 \frac{k_r^4}{k_u^4} - B_0^2 \frac{k_s^4}{k_u^4} + 2 (B_{xs}B_{ys} - B_{xc}B_{yc}) \frac{k_r^2 k_s^2}{k_u^4} \cos(\delta) + \frac{1}{2} \left(B_{yc}^2 + B_{xc}^2 \right) \frac{k_s^2}{k_u^2} (\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})]$$
$$+ k_u \sqrt{2} y_{0L} [\frac{1}{2} \left(B_{yc}^2 - B_{xc}^2 \right) \frac{k_s^2}{k_u^2} (\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})] \} \tag{174}$$

$$\langle v_{yL} \rangle = \frac{1}{2\sqrt{2}} \frac{q^2}{\gamma^2 m^2 k_u v_{z(0)}} z$$
$$\times \{ (B_{xc}B_{ys} - B_{yc}B_{xs}) \frac{k_r k_s^2}{k_u^3} \sin(\delta)$$
$$+ k_u \sqrt{2} y_{0L} [-B_0^2 \frac{k_r^4}{k_u^4} - B_0^2 \frac{k_s^4}{k_u^4} + 2 (B_{xs}B_{ys} + B_{xc}B_{yc}) \frac{k_r^2 k_s^2}{k_u^4} \cos(\delta) + \frac{1}{2} \left(B_{yc}^2 + B_{xc}^2 \right) \frac{k_s^2}{k_u^2} (\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})]$$
$$+ k_u \sqrt{2} x_{0L} [\frac{1}{2} \left(B_{yc}^2 - B_{xc}^2 \right) \frac{k_s^2}{k_u^2} (\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})] \} \tag{175}$$

Since

$$v_{xL} = \frac{dx_L}{dz} v_z \tag{176}$$

$$v_{yL} = \frac{dy_L}{dz} v_z \tag{177}$$

we can easily find the slope of the trajectory in the undulator.

$$
\begin{aligned}
\left\langle \frac{dx_L}{dz} \right\rangle &= \frac{1}{2\sqrt{2}} \frac{q^2}{\gamma^2 m^2 k_u v_{z(0)}^2} z \\
&\times\{(B_{xc}B_{ys} + B_{yc}B_{xs}) \frac{k_r k_s^2}{k_u^3} \sin(\delta) \\
&+ k_u \sqrt{2} x_{0L} [-B_0^2 \frac{k_r^4}{k_u^4} - B_0^2 \frac{k_s^4}{k_u^4} + 2\left(B_{xs}B_{ys} - B_{xc}B_{yc}\right) \frac{k_r^2 k_s^2}{k_u^4} \cos(\delta) + \frac{1}{2}\left(B_{yc}^2 + B_{xc}^2\right) \frac{k_s^2}{k_u^2} (\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})] \\
&+ k_u \sqrt{2} y_{0L} [\frac{1}{2}\left(B_{yc}^2 - B_{xc}^2\right) \frac{k_s^2}{k_u^2} (\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})]\} \qquad (178)
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \frac{dy_L}{dz} \right\rangle &= \frac{1}{2\sqrt{2}} \frac{q^2}{\gamma^2 m^2 k_u v_{z(0)}^2} z \\
&\times\{(B_{xc}B_{ys} - B_{yc}B_{xs}) \frac{k_r k_s^2}{k_u^3} \sin(\delta) \\
&+ k_u \sqrt{2} y_{0L} [-B_0^2 \frac{k_r^4}{k_u^4} - B_0^2 \frac{k_s^4}{k_u^4} + 2\left(B_{xs}B_{ys} + B_{xc}B_{yc}\right) \frac{k_r^2 k_s^2}{k_u^4} \cos(\delta) + \frac{1}{2}\left(B_{yc}^2 + B_{xc}^2\right) \frac{k_s^2}{k_u^2} (\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})] \\
&+ k_u \sqrt{2} x_{0L} [\frac{1}{2}\left(B_{yc}^2 - B_{xc}^2\right) \frac{k_s^2}{k_u^2} (\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})]\} \qquad (179)
\end{aligned}
$$

4.2 Parameters k_x and k_y

If the transverse behavior of the field is measured after the undulator is assembled, then in the laboratory frame, one determines new parameters, say k_x and k_y, describing the field instead of k_r and k_s. The parameters are related, however. Consider the undulator in planar vertical field mode and at the maximum K value. In this case $\Delta_{13} = 0$, $Z_{13} = 0$, $\Delta_{24} = 0$, and $Z_{24} = 0$. The scalar potential in the undulator frame is

$$
\begin{aligned}
\phi &= 2\phi_{0Q} \cos(k_s y) \sinh(k_r x) \cos(k_u z) \\
&+ 2\phi_{0Q} \cos(k_s x) \sinh(k_r y) \cos(k_u z) \qquad (180)
\end{aligned}
$$

In the laboratory frame, the scalar potential is

$$
\begin{aligned}
\phi &= 2\phi_{0Q} \cos\left(k_s \frac{1}{\sqrt{2}}(-x_L + y_L)\right) \sinh\left(k_r \frac{1}{\sqrt{2}}(x_L + y_L)\right) \cos(k_u z) \\
&+ 2\phi_{0Q} \cos\left(k_s \frac{1}{\sqrt{2}}(x_L + y_L)\right) \sinh\left(k_r \frac{1}{\sqrt{2}}(-x_L + y_L)\right) \cos(k_u z) \qquad (181)
\end{aligned}
$$

The vertical field is $B_y = \partial_y \phi$:

$$
\begin{aligned}
B_y &= -2\phi_{0Q} k_s \frac{1}{\sqrt{2}} \sin\left(k_s \frac{1}{\sqrt{2}}(-x_L + y_L)\right) \sinh\left(k_r \frac{1}{\sqrt{2}}(x_L + y_L)\right) \cos(k_u z) \\
&+ 2\phi_{0Q} k_r \frac{1}{\sqrt{2}} \cos\left(k_s \frac{1}{\sqrt{2}}(-x_L + y_L)\right) \cosh\left(k_r \frac{1}{\sqrt{2}}(x_L + y_L)\right) \cos(k_u z) \\
&- 2\phi_{0Q} k_s \frac{1}{\sqrt{2}} \sin\left(k_s \frac{1}{\sqrt{2}}(x_L + y_L)\right) \sinh\left(k_r \frac{1}{\sqrt{2}}(-x_L + y_L)\right) \cos(k_u z) \\
&+ 2\phi_{0Q} k_r \frac{1}{\sqrt{2}} \cos\left(k_s \frac{1}{\sqrt{2}}(x_L + y_L)\right) \cosh\left(k_r \frac{1}{\sqrt{2}}(-x_L + y_L)\right) \cos(k_u z) \qquad (182)
\end{aligned}
$$

On the horizontal midplane, $y_L = 0$, and the field is

$$\begin{aligned} B_y(x_L,0,z) &= 4\phi_{0Q}k_s\frac{1}{\sqrt{2}}\sin\left(k_s\frac{1}{\sqrt{2}}x_L\right)\sinh\left(k_r\frac{1}{\sqrt{2}}x_L\right)\cos(k_u z) \\ &\quad +4\phi_{0Q}k_r\frac{1}{\sqrt{2}}\cos\left(k_s\frac{1}{\sqrt{2}}x_L\right)\cosh\left(k_r\frac{1}{\sqrt{2}}x_L\right)\cos(k_u z) \end{aligned} \tag{183}$$

Near the origin, the field can be approximated as

$$\begin{aligned} B_y(x_L,0,z) &= 4\phi_{0Q}k_s\frac{1}{\sqrt{2}}\left(k_s\frac{1}{\sqrt{2}}x_L\right)\left(k_r\frac{1}{\sqrt{2}}x_L\right)\cos(k_u z) \\ &\quad +4\phi_{0Q}k_r\frac{1}{\sqrt{2}}[1-\frac{1}{2}\left(k_s\frac{1}{\sqrt{2}}x_L\right)^2][1+\frac{1}{2}\left(k_r\frac{1}{\sqrt{2}}x_L\right)^2]\cos(k_u z) \end{aligned} \tag{184}$$

which simplifies to

$$B_y(x_L,0,z) = 4\phi_{0Q}\frac{k_r}{\sqrt{2}}[1+\frac{1}{2}(\frac{k_s^2+k_r^2}{2})x_L^2]\cos(k_u z) \tag{185}$$

The field varies in the horizontal direction as

$$B_y(x_L,0,z) \sim \cosh(k_x x_L) \tag{186}$$

with an effective parameter

$$k_x^2 = \frac{k_s^2+k_r^2}{2} \tag{187}$$

On the vertical midplane with $x_L = 0$, the vertical field is

$$\begin{aligned} B_y &= -4\phi_{0Q}k_s\frac{1}{\sqrt{2}}\sin\left(k_s\frac{1}{\sqrt{2}}y_L\right)\sinh\left(k_r\frac{1}{\sqrt{2}}y_L\right)\cos(k_u z) \\ &\quad +4\phi_{0Q}k_r\frac{1}{\sqrt{2}}\cos\left(k_s\frac{1}{\sqrt{2}}y_L\right)\cosh\left(k_r\frac{1}{\sqrt{2}}y_L\right)\cos(k_u z) \end{aligned} \tag{188}$$

Near the origin the field can be approximated as

$$\begin{aligned} B_y(0,y_L,z) &= -4\phi_{0Q}k_s\frac{1}{\sqrt{2}}\left(k_s\frac{1}{\sqrt{2}}y_L\right)\left(k_r\frac{1}{\sqrt{2}}y_L\right)\cos(k_u z) \\ &\quad +4\phi_{0Q}k_r\frac{1}{\sqrt{2}}[1-\frac{1}{2}\left(k_s\frac{1}{\sqrt{2}}y_L\right)^2][1+\frac{1}{2}\left(k_r\frac{1}{\sqrt{2}}y_L\right)^2]\cos(k_u z) \end{aligned} \tag{189}$$

which simplifies to

$$B_y(0,y_L,z) = 4\phi_{0Q}k_r\frac{1}{\sqrt{2}}[1-\frac{1}{2}(\frac{3k_s^2-k_r^2}{2})y_L^2]\cos(k_u z) \tag{190}$$

The field varies in the vertical direction as

$$B_y(0,y_L,z) \sim \cos(k_y y_L) \tag{191}$$

with an effective parameter

$$k_y^2 = (\frac{3k_s^2-k_r^2}{2}) \tag{192}$$

We can find k_r and k_s from k_x and k_y. We see that

$$k_x^2 + k_y^2 = 2k_s^2 \tag{193}$$

$$k_y^2 - 3k_x^2 = -2k_r^2 \tag{194}$$

So

$$k_s^2 = \frac{k_x^2 + k_y^2}{2} \tag{195}$$

$$k_r^2 = \frac{3k_x^2 - k_y^2}{2} \tag{196}$$

5 Deflection Of The Beam

We can easily find the slope of the trajectory when a particle leaves the undulator. If the undulator has length L, using equations 178 and 179, the slope of the trajectory in the laboratory frame when the beam leaves the undulator is

$$\begin{aligned}\left\langle \frac{dx_L}{dz} \right\rangle &= \frac{1}{2\sqrt{2}} \frac{q^2}{\gamma^2 m^2 k_u v_{z0}^2} L \\ &\times \{ (B_{xc}B_{ys} + B_{yc}B_{xs}) \frac{k_r k_s^2}{k_u^3} \sin(\delta) \\ &+ k_u\sqrt{2} x_{0L} [-B_0^2 \frac{k_r^4}{k_u^4} - B_0^2 \frac{k_s^4}{k_u^4} + 2\left(B_{xs}B_{ys} - B_{xc}B_{yc}\right) \frac{k_r^2 k_s^2}{k_u^4} \cos(\delta) + \frac{1}{2}\left(B_{yc}^2 + B_{xc}^2\right) \frac{k_s^2}{k_u^2} (\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})] \\ &+ k_u\sqrt{2} y_{0L} [\frac{1}{2}\left(B_{yc}^2 - B_{xc}^2\right) \frac{k_s^2}{k_u^2} (\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})] \}\end{aligned} \tag{197}$$

$$\begin{aligned}\left\langle \frac{dy_L}{dz} \right\rangle &= \frac{1}{2\sqrt{2}} \frac{q^2}{\gamma^2 m^2 k_u v_{z0}^2} L \\ &\times \{ (B_{xc}B_{ys} - B_{yc}B_{xs}) \frac{k_r k_s^2}{k_u^3} \sin(\delta) \\ &+ k_u\sqrt{2} y_{0L} [-B_0^2 \frac{k_r^4}{k_u^4} - B_0^2 \frac{k_s^4}{k_u^4} + 2\left(B_{xs}B_{ys} + B_{xc}B_{yc}\right) \frac{k_r^2 k_s^2}{k_u^4} \cos(\delta) + \frac{1}{2}\left(B_{yc}^2 + B_{xc}^2\right) \frac{k_s^2}{k_u^2} (\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})] \\ &+ k_u\sqrt{2} x_{0L} [\frac{1}{2}\left(B_{yc}^2 - B_{xc}^2\right) \frac{k_s^2}{k_u^2} (\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})] \}\end{aligned} \tag{198}$$

We can express these equations in terms of the parameters giving the row positions using the following equations

$$\begin{aligned}B_{xc}B_{ys} \pm B_{yc}B_{xs} &= B_0^2 [\cos\left(k_u \frac{\Delta_{13}}{2}\right) \sin\left(k_u \frac{\Delta_{24}}{2}\right) \pm \cos\left(k_u \frac{\Delta_{24}}{2}\right) \sin\left(k_u \frac{\Delta_{13}}{2}\right)] \\ &= B_0^2 \sin\left(k_u \frac{\Delta_{24} \pm \Delta_{13}}{2}\right)\end{aligned} \tag{199}$$

$$\begin{aligned}B_{xs}B_{ys} \mp B_{xc}B_{yc} &= B_0^2 [\sin\left(k_u \frac{\Delta_{13}}{2}\right) \sin\left(k_u \frac{\Delta_{24}}{2}\right) \mp \cos\left(k_u \frac{\Delta_{13}}{2}\right) \cos\left(k_u \frac{\Delta_{24}}{2}\right)] \\ &= \mp B_0^2 \cos\left(k_u \frac{\Delta_{24} \pm \Delta_{13}}{2}\right)\end{aligned} \tag{200}$$

$$
\begin{aligned}
B_{yc}^2 + B_{xc}^2 &= B_0^2[\cos^2\left(k_u\frac{\Delta_{24}}{2}\right) + \cos^2\left(k_u\frac{\Delta_{13}}{2}\right)] \\
&= B_0^2\{\frac{1}{2}[1+\cos(k_u\Delta_{24})] + \frac{1}{2}[1+\cos(k_u\Delta_{13})]\} \\
&= B_0^2\{1 + \frac{1}{2}[\cos(k_u\Delta_{24}) + \cos(k_u\Delta_{13})]\} \\
&= B_0^2[1 + \cos\left(k_u\frac{\Delta_{24}+\Delta_{13}}{2}\right)\cos\left(k_u\frac{\Delta_{24}-\Delta_{13}}{2}\right)]
\end{aligned} \tag{201}
$$

$$
\begin{aligned}
B_{yc}^2 - B_{xc}^2 &= B_0^2[\cos^2\left(k_u\frac{\Delta_{24}}{2}\right) - \cos^2\left(k_u\frac{\Delta_{13}}{2}\right)] \\
&= B_0^2\{\frac{1}{2}[1+\cos(k_u\Delta_{24})] - \frac{1}{2}[1+\cos(k_u\Delta_{13})]\} \\
&= B_0^2\{\frac{1}{2}[\cos(k_u\Delta_{24}) - \cos(k_u\Delta_{13})]\} \\
&= -B_0^2\sin\left(k_u\frac{\Delta_{24}+\Delta_{13}}{2}\right)\sin\left(k_u\frac{\Delta_{24}-\Delta_{13}}{2}\right)
\end{aligned} \tag{202}
$$

The equations for the slopes of the exit trajectories become

$$
\begin{aligned}
\left\langle\frac{dx_L}{dz}\right\rangle &= \frac{q^2B_0^2}{2\gamma^2m^2k_uv_{z0}^2}L \\
&\times\{\frac{1}{\sqrt{2}}\sin\left(k_u\frac{\Delta_{24}+\Delta_{13}}{2}\right)\frac{k_rk_s^2}{k_u^3}\sin(\delta) \\
&-k_ux_{0L}[\frac{k_r^4}{k_u^4} + \frac{1}{2}\frac{k_s^4}{k_u^4} - \frac{1}{2}\frac{k_r^2k_s^2}{k_u^4} + 2\cos\left(k_u\frac{\Delta_{24}+\Delta_{13}}{2}\right)\frac{k_r^2k_s^2}{k_u^4}\cos(\delta) \\
&-\frac{1}{2}\cos\left(k_u\frac{\Delta_{24}+\Delta_{13}}{2}\right)\cos\left(k_u\frac{\Delta_{24}-\Delta_{13}}{2}\right)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})] \\
&-k_uy_{0L}[\frac{1}{2}\sin\left(k_u\frac{\Delta_{24}+\Delta_{13}}{2}\right)\sin\left(k_u\frac{\Delta_{24}-\Delta_{13}}{2}\right)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})]\}
\end{aligned} \tag{203}
$$

$$
\begin{aligned}
\left\langle\frac{dy_L}{dz}\right\rangle &= \frac{q^2B_0^2}{2\gamma^2m^2k_uv_{z0}^2}L \\
&\times\{\frac{1}{\sqrt{2}}\sin\left(k_u\frac{\Delta_{24}-\Delta_{13}}{2}\right)\frac{k_rk_s^2}{k_u^3}\sin(\delta) \\
&-k_uy_{0L}[\frac{k_r^4}{k_u^4} + \frac{1}{2}\frac{k_s^4}{k_u^4} - \frac{1}{2}\frac{k_r^2k_s^2}{k_u^4} - 2\cos\left(k_u\frac{\Delta_{24}-\Delta_{13}}{2}\right)\frac{k_r^2k_s^2}{k_u^4}\cos(\delta) \\
&-\frac{1}{2}\cos\left(k_u\frac{\Delta_{24}+\Delta_{13}}{2}\right)\cos\left(k_u\frac{\Delta_{24}-\Delta_{13}}{2}\right)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})] \\
&-k_ux_{0L}[\frac{1}{2}\sin\left(k_u\frac{\Delta_{24}+\Delta_{13}}{2}\right)\sin\left(k_u\frac{\Delta_{24}-\Delta_{13}}{2}\right)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})]\}
\end{aligned} \tag{204}
$$

The exit slopes each have three contributions. There is a term that does not depend on where the particle enters the undulator. There is a focusing term that depends on the coordinate whose slope is being calculated, and there is a skew focusing term that depends on the orthogonal coordinate.

We write the exit slopes as

$$\left\langle \frac{dx_L}{dz} \right\rangle = x'_{L0} - \frac{x_{0L}}{f_x} - \frac{y_{0L}}{s_x} \tag{205}$$

$$\left\langle \frac{dy_L}{dz} \right\rangle = y'_{L0} - \frac{y_{0L}}{f_y} - \frac{x_{0L}}{s_y} \tag{206}$$

In these expressions, f_x and f_y correspond to focal lengths. In an analogous way, we define skew focal lengths s_x and s_y from these equations.

Note that if $k_s = 0$ and $k_r = k_u$, we have

$$\left\langle \frac{dx_L}{dz} \right\rangle = -\frac{q^2 B_0^2 L}{2\gamma^2 m^2 v_{z0}^2} x_{L0} \tag{207}$$

$$\left\langle \frac{dy_L}{dz} \right\rangle = -\frac{q^2 B_0^2 L}{2\gamma^2 m^2 v_{z0}^2} y_{L0} \tag{208}$$

which agrees with a previous result[5]. We see that the focal length of the undulator is

$$f_x = f_y = \frac{2\gamma^2 m^2 v_{z0}^2}{q^2 B_0^2 L} \tag{209}$$

The focal length is the same in both the x and y directions. If $k_s = 0$, the focal length is independent of the mode of the undulator and is independent of the K value.

6 The Primary Modes

6.1 Undulator Configuration In The Primary Modes

The Delta undulator is mainly used in four primary modes: linear polarization vertical magnetic field, linear polarization horizontal magnetic field, circular polarization right hand magnetic field, and circular polarization left hand magnetic field. In these modes the K value is set by making equal shifts of the 1-3 and 2-4 crossed planar undulators.

$$\Delta_{13} = \Delta_{24} \equiv \Delta \tag{210}$$

The mode is set by shifting the crossed planar undulators relative to each other. This is equivalent to changing the value of δ[6].

For given magnetic field parameters, we can determine the quadrant positions. There are three adjustable magnetic field parameters, Δ_{13}, Δ_{24}, and δ, so we need one additional constraint equation in order to determine the four quadrant positions. We choose the extra constraint to be that the average z-position of the quadrants is zero.

$$z_{01} + z_{02} + z_{03} + z_{04} = 0 \tag{211}$$

The other three equations are

$$z_{01} - z_{03} = \Delta_{13} \tag{212}$$

$$z_{02} - z_{04} = \Delta_{24} \tag{213}$$

$$z_{01} - z_{02} + z_{03} - z_{04} = \frac{2}{k_u}\delta \tag{214}$$

[5] Z. Wolf, "Beam Trajectories In The Delta Undulator I", LCLS-TN-15-3, March, 2015.
[6] Z. Wolf, H.-D. Nuhn, "Setting The K Value And Polarization Mode Of The Delta Undulator", LCLS-TN-14-2, September, 2014.

The solution for the row positions in terms of the field specifying parameters is

$$z_{01} = \frac{1}{2}\left(\frac{\delta}{k_u} + \Delta_{13}\right) \quad (215)$$

$$z_{02} = \frac{1}{2}\left(-\frac{\delta}{k_u} + \Delta_{24}\right) \quad (216)$$

$$z_{03} = \frac{1}{2}\left(\frac{\delta}{k_u} - \Delta_{13}\right) \quad (217)$$

$$z_{04} = \frac{1}{2}\left(-\frac{\delta}{k_u} - \Delta_{24}\right) \quad (218)$$

The parameters determining the field in the primary modes are[7]

	Δ_{13}	Δ_{24}	Z_{13}	Z_{24}	δ
Linear Vertical	Δ	Δ	0	0	0
Linear Horizontal	Δ	Δ	$\lambda_u/4$	$-\lambda_u/4$	π
Circular Right	Δ	Δ	$-\lambda_u/8$	$\lambda_u/8$	$-\pi/2$
Circular Left	Δ	Δ	$\lambda_u/8$	$-\lambda_u/8$	$\pi/2$

The corresponding quadrant positions are

	z_{01}	z_{02}	z_{03}	z_{04}
Linear Vertical	$\Delta/2$	$\Delta/2$	$-\Delta/2$	$-\Delta/2$
Linear Horizontal	$\lambda_u/4 + \Delta/2$	$-\lambda_u/4+\Delta/2$	$\lambda_u/4-\Delta/2$	$-\lambda_u/4-\Delta/2$
Circular Right	$-\lambda_u/8 + \Delta/2$	$\lambda_u/8+\Delta/2$	$-\lambda_u/8-\Delta/2$	$\lambda_u/8-\Delta/2$
Circular Left	$\lambda_u/8 + \Delta/2$	$-\lambda_u/8+\Delta/2$	$\lambda_u/8-\Delta/2$	$-\lambda_u/8-\Delta/2$

6.2 Exit Trajectory Slope In The Primary Modes

In the primary modes with $\Delta_{13} = \Delta_{24} \equiv \Delta$, the formulas giving the exit slope of the trajectory simplify to

$$\begin{aligned}\left\langle \frac{dx_L}{dz} \right\rangle = & \frac{q^2 B_0^2}{2\gamma^2 m^2 k_u v_{z0}^2} L\{ \frac{1}{\sqrt{2}} \sin(k_u \Delta) \frac{k_r k_s^2}{k_u^3} \sin(\delta) \\ & -k_u x_{0L}[\frac{k_r^4}{k_u^4} + \frac{1}{2}\frac{k_s^4}{k_u^4} - \frac{1}{2}\frac{k_r^2 k_s^2}{k_u^4} + 2\cos(k_u\Delta)\frac{k_r^2 k_s^2}{k_u^4}\cos(\delta) \\ & -\frac{1}{2}\cos(k_u\Delta)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})]\}\end{aligned} \quad (219)$$

$$\begin{aligned}\left\langle \frac{dy_L}{dz} \right\rangle = & \frac{q^2 B_0^2}{2\gamma^2 m^2 k_u v_{z0}^2} L\{ \\ & -k_u y_{0L}[\frac{k_r^4}{k_u^4} + \frac{1}{2}\frac{k_s^4}{k_u^4} - \frac{1}{2}\frac{k_r^2 k_s^2}{k_u^4} - 2\frac{k_r^2 k_s^2}{k_u^4}\cos(\delta) - \frac{1}{2}\cos(k_u\Delta)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})]\}\end{aligned} \quad (220)$$

Note that there are no skew terms with this method of setting the Delta undulator.

[7] Z. Wolf, H.-D. Nuhn, "Setting The K Value And Polarization Mode Of The Delta Undulator", LCLS-TN-14-2, September, 2014.

6.3 Numerical Calculations

In this section we calculate the exit slope of the trajectory in the four primary modes of the Delta undulator. We make all calculations with

$$\gamma = 10000 \tag{221}$$

corresponding to an energy of 5.11 GeV.

In these calculations, we use

$$k_s = 186 \text{ 1/m} \tag{222}$$

This is a value measured on a Delta quadrant array[8]. Using the constraint $k_r^2 = k_s^2 + k_u^2$, we find

$$k_r = 270 \text{ 1/m} \tag{223}$$

In linear polarization vertical field mode, we use k_s and k_r to calculate k_x and k_y. We find

$$k_x = 232 \text{ 1/m} \tag{224}$$
$$k_y = 124 \text{ 1/m} \tag{225}$$

These values are in good agreement with results from a Delta undulator simulation of $k_x = 222$ 1/m and $k_y = 129$ 1/m[9].

In the calculations that follow, we use the measured field in the undulator in linear polarization vertical field mode at maximum K to determine the quadrant field parameter B_0. The measured undulator field is approximately 1.2 T. Using equation185, the peak field in the undulator is $4\phi_{0Q}k_r/\sqrt{2} = 1.2$ T. Since $B_0 = 2\phi_{0Q}k_u$, we have

$$B_0 = 0.61 \text{ T} \tag{226}$$

We take the length of the undulator to be the 96 periods in the center of the undulator, or

$$L = 3.1 \text{ m} \tag{227}$$

6.4 Simulations

In order to check the formulas in this note, an independent simulation was performed. The fields from each quadrant were calculated including the field rolloff as explained above. The fields from each quadrant were shifted according to the z_{0i} parameters and were then summed. This allowed the calculation of the field at any point in the undulator. A particle was sent into this field at a given position and with a given transverse velocity. The Lorentz force equations were used to successively step along the trajectories. The field at the particle location was used in the calculations, so second order (and higher) effects are included. In the following discussion, these trajectories are compared to the formulas in this note.

6.5 Linear Polarization Vertical Magnetic Field

In the linear polarization vertical magnetic field mode,

$$\delta = 0 \tag{228}$$

[8] Yurii Levashov, private communication.
[9] Alexander Temnykh, private communication.

In this case

$$\left\langle \frac{dx_L}{dz} \right\rangle = \frac{q^2 B_0^2}{2\gamma^2 m^2 k_u v_{z0}^2} z\{-k_u x_{0L}[\frac{k_r^4}{k_u^4} + \frac{1}{2}\frac{k_s^4}{k_u^4} - \frac{1}{2}\frac{k_r^2 k_s^2}{k_u^4} + 2\cos(k_u\Delta)\frac{k_r^2 k_s^2}{k_u^4} - \frac{1}{2}\cos(k_u\Delta)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})]\} \quad (229)$$

$$\left\langle \frac{dy_L}{dz} \right\rangle = \frac{q^2 B_0^2}{2\gamma^2 m^2 k_u v_{z0}^2} z\{-k_u y_{0L}[\frac{k_r^4}{k_u^4} + \frac{1}{2}\frac{k_s^4}{k_u^4} - \frac{5}{2}\frac{k_r^2 k_s^2}{k_u^4} - \frac{1}{2}\cos(k_u\Delta)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})]\} \quad (230)$$

Using the parameter values given above, the trajectories from the simulation are compared to these formulas in figures 3 and 4. In these figures, the row shift Δ is zero corresponding to the maximum K value. The particle enters the undulator at $x_{0L} = 10$ μm, $y_{0L} = 10$ μm with zero initial slope.

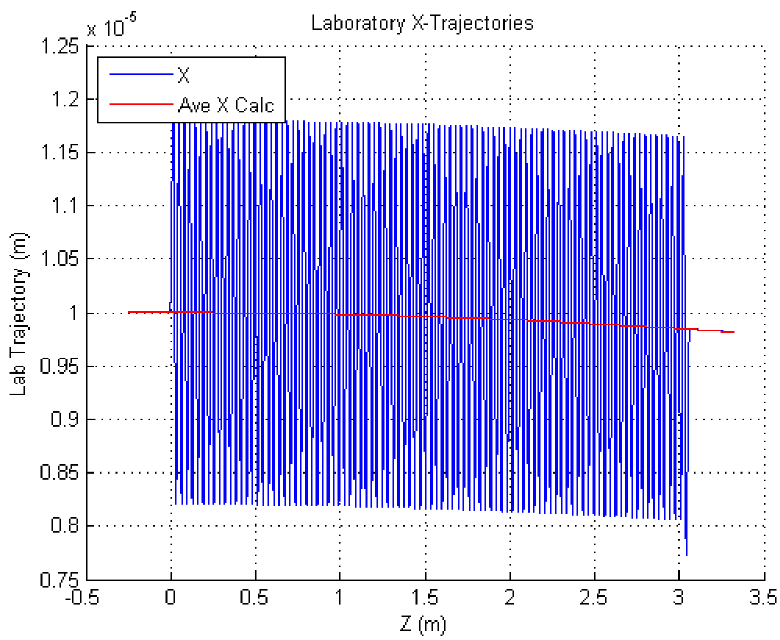

Figure 3: X-trajectory calculated by the simulation and compared to the average x-trajectory calculated using the equations in this note. Linear polarization vertical magnetic field mode.

The inverse focal length is given by

$$\left\langle \frac{dx_L}{dz} \right\rangle_{z=L} = -x_{0L}\frac{1}{f_x} \quad (231)$$

$$\frac{1}{f_x} = \frac{q^2 B_0^2 L}{2\gamma^2 m^2 v_{z0}^2}[\frac{k_r^4}{k_u^4} + \frac{1}{2}\frac{k_s^4}{k_u^4} - \frac{1}{2}\frac{k_r^2 k_s^2}{k_u^4} + 2\cos(k_u\Delta)\frac{k_r^2 k_s^2}{k_u^4} - \frac{1}{2}\cos(k_u\Delta)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})] \quad (232)$$

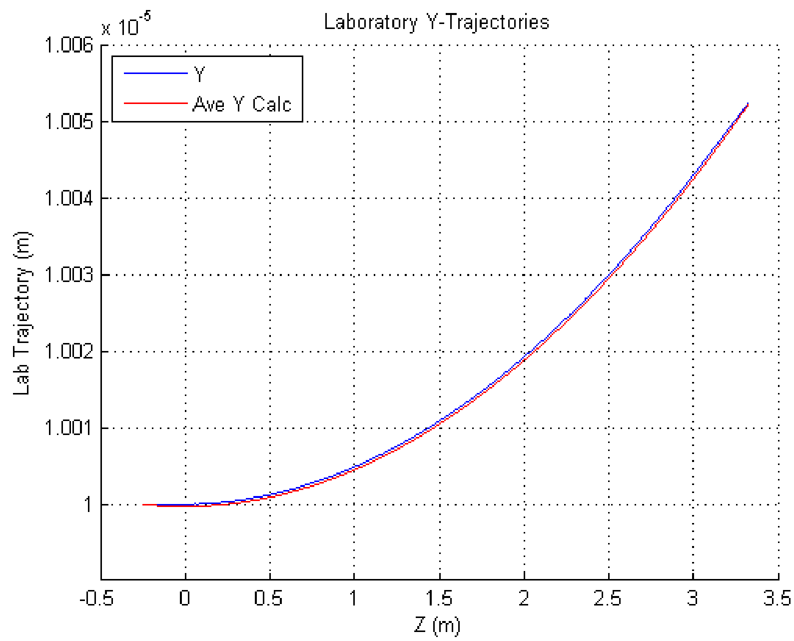

Figure 4: Y-trajectory calculated by the simulation and compared to the average y-trajectory calculated using the equations in this note. Linear polarization vertical magnetic field mode.

$$\left\langle \frac{dy_L}{dz} \right\rangle_{z=L} = -y_{0L}\frac{1}{f_y} \tag{233}$$

$$\frac{1}{f_y} = \frac{q^2 B_0^2 L}{2\gamma^2 m^2 v_{z0}^2}\left[\frac{k_r^4}{k_u^4} + \frac{1}{2}\frac{k_s^4}{k_u^4} - \frac{5}{2}\frac{k_r^2 k_s^2}{k_u^4} - \frac{1}{2}\cos\left(k_u\Delta\right)\frac{k_s^2}{k_u^2}\left(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2}\right)\right] \tag{234}$$

The inverse focal length is plotted as a function of Δ in figure 5.

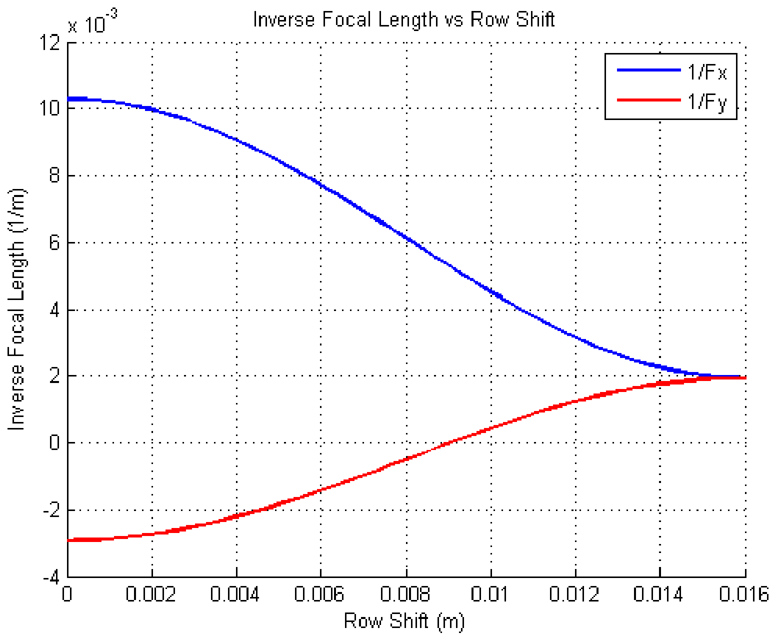

Figure 5: Inverse focal length as a function of Δ in linear polarization vertical magnetic field mode.

6.6 Linear Polarization Horizontal Magnetic Field

In the linear polarization horizontal magnetic field mode,

$$\delta = \pi \tag{235}$$

In this case

$$\begin{aligned}\left\langle \frac{dx_L}{dz} \right\rangle &= \frac{q^2 B_0^2}{2\gamma^2 m^2 k_u v_{z0}^2} z\{-k_u x_{0L}[\frac{k_r^4}{k_u^4} + \frac{1}{2}\frac{k_s^4}{k_u^4} - \frac{1}{2}\frac{k_r^2 k_s^2}{k_u^4} - 2\cos(k_u\Delta)\frac{k_r^2 k_s^2}{k_u^4} \\ &\quad -\frac{1}{2}\cos(k_u\Delta)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})]\}\end{aligned} \tag{236}$$

$$\left\langle \frac{dy_L}{dz} \right\rangle = \frac{q^2 B_0^2}{2\gamma^2 m^2 k_u v_{z0}^2} z\{-k_u y_{0L}[\frac{k_r^4}{k_u^4} + \frac{1}{2}\frac{k_s^4}{k_u^4} + \frac{3}{2}\frac{k_r^2 k_s^2}{k_u^4} - \frac{1}{2}\cos(k_u\Delta)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})]\} \tag{237}$$

Using the parameter values given above, the trajectories from the simulation are compared to these formulas in figures 6 and 7. In these figures, the row shift Δ is zero corresponding to the maximum K value. The particle enters the undulator at $x_{0L} = 10$ μm, $y_{0L} = 10$ μm with zero initial slope.

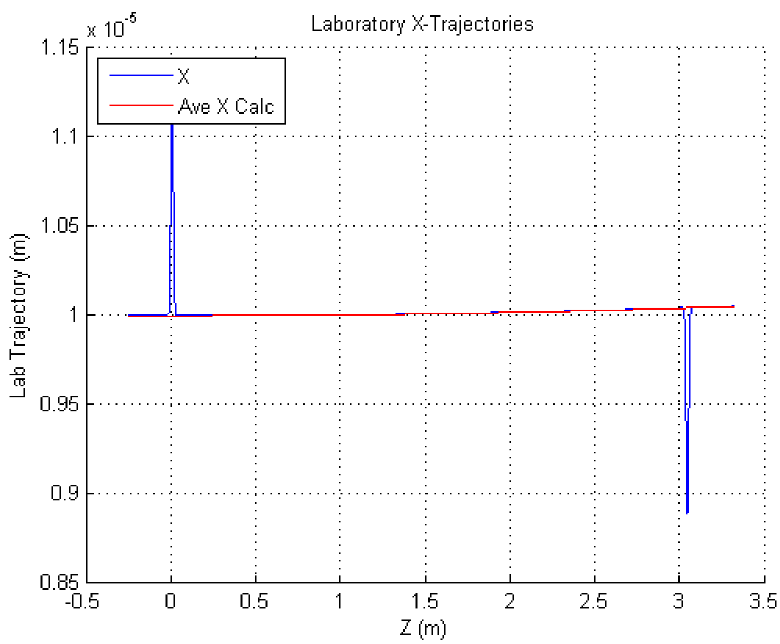

Figure 6: X-trajectory calculated by the simulation and compared to the average x-trajectory calculated using the equations in this note. Linear polarization horizontal magnetic field mode.

The inverse focal length is given by

$$\left\langle \frac{dx_L}{dz} \right\rangle_{z=L} = -x_{0L}\frac{1}{f_x} \tag{238}$$

$$\begin{aligned}\frac{1}{f_x} &= \frac{q^2 B_0^2 L}{2\gamma^2 m^2 v_{z0}^2}[\frac{k_r^4}{k_u^4} + \frac{1}{2}\frac{k_s^4}{k_u^4} - \frac{1}{2}\frac{k_r^2 k_s^2}{k_u^4} - 2\cos(k_u\Delta)\frac{k_r^2 k_s^2}{k_u^4} \\ &\quad -\frac{1}{2}\cos(k_u\Delta)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})]\end{aligned} \tag{239}$$

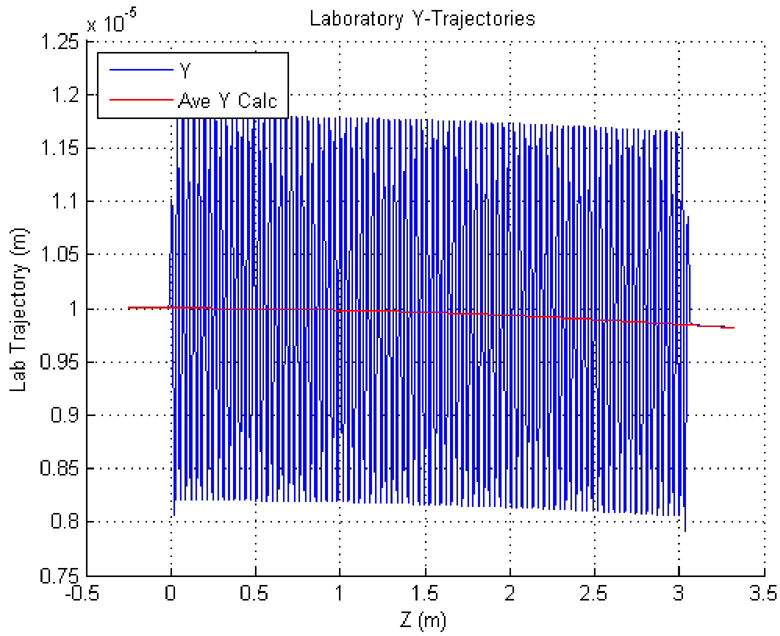

Figure 7: Y-trajectory calculated by the simulation and compared to the average y-trajectory calculated using the equations in this note. Linear polarization horizontal magnetic field mode.

$$\left\langle \frac{dy_L}{dz} \right\rangle_{z=L} = -y_{0L}\frac{1}{f_y} \tag{240}$$

$$\frac{1}{f_y} = \frac{q^2 B_0^2 L}{2\gamma^2 m^2 v_{z0}^2}[\frac{k_r^4}{k_u^4} + \frac{1}{2}\frac{k_s^4}{k_u^4} + \frac{3}{2}\frac{k_r^2 k_s^2}{k_u^4} - \frac{1}{2}\cos(k_u\Delta)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})] \tag{241}$$

The inverse focal length is plotted as a function of Δ in figure 8.

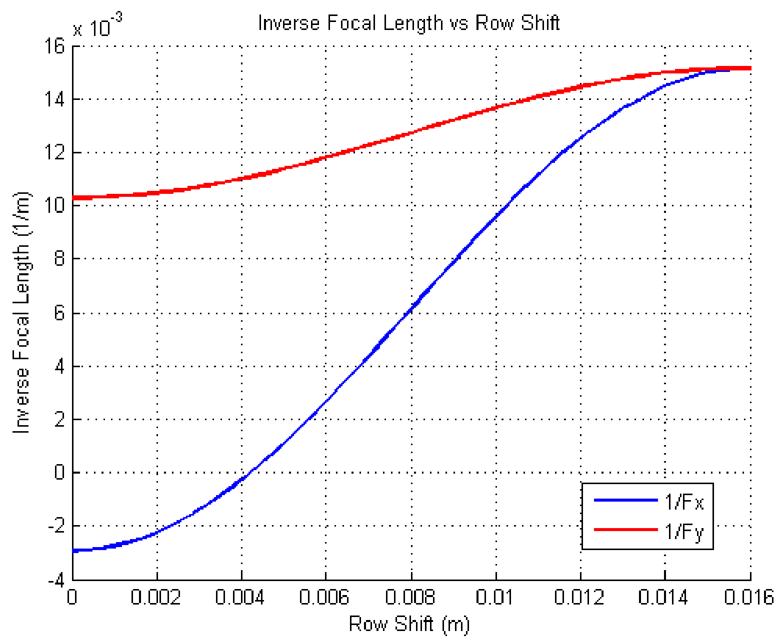

Figure 8: Inverse focal length as a function of Δ in linear polarization horizontal magnetic field mode.

6.7 Circular Polarization Right Hand Magnetic Field

In the circular polarization right hand magnetic field mode,

$$\delta = -\frac{\pi}{2} \tag{242}$$

In this case

$$\begin{aligned} \left\langle \frac{dx_L}{dz} \right\rangle &= \frac{q^2 B_0^2}{2\gamma^2 m^2 k_u v_{z0}^2} z\{-\frac{1}{\sqrt{2}} \sin(k_u \Delta) \frac{k_r k_s^2}{k_u^3} \\ &\quad -k_u x_{0L}[\frac{k_r^4}{k_u^4} + \frac{1}{2}\frac{k_s^4}{k_u^4} - \frac{1}{2}\frac{k_r^2 k_s^2}{k_u^4} - \frac{1}{2}\cos(k_u\Delta)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})]\} \end{aligned} \tag{243}$$

$$\begin{aligned} \left\langle \frac{dy_L}{dz} \right\rangle &= \frac{q^2 B_0^2}{2\gamma^2 m^2 k_u v_{z0}^2} z\{ \\ &\quad -k_u y_{0L}[\frac{k_r^4}{k_u^4} + \frac{1}{2}\frac{k_s^4}{k_u^4} - \frac{1}{2}\frac{k_r^2 k_s^2}{k_u^4} - \frac{1}{2}\cos(k_u\Delta)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})]\} \end{aligned} \tag{244}$$

Using the parameter values given above, the trajectories from the simulation are compared to these formulas in figures 9 and 10. In these figures, the row shift Δ is zero corresponding to the maximum K value. The particle enters the undulator at $x_{0L} = 10\ \mu$m, $y_{0L} = 10\ \mu$m with zero initial slope.

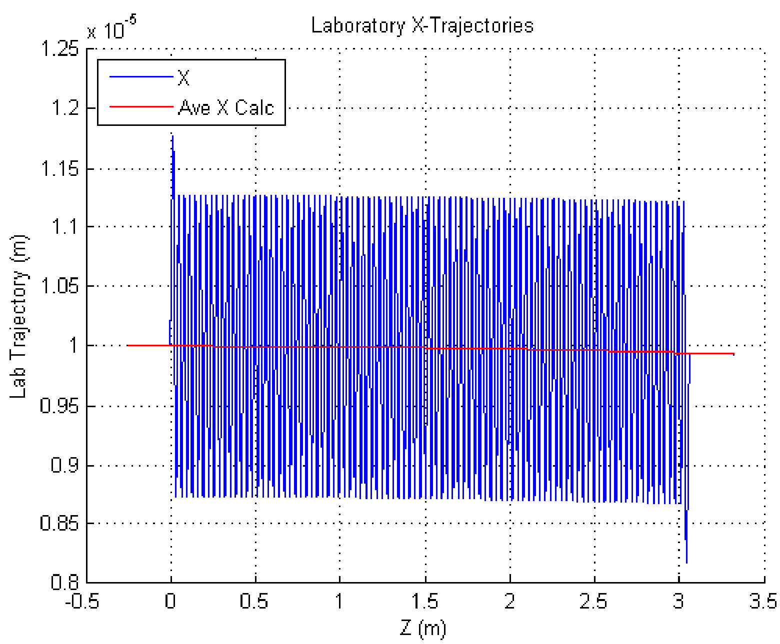

Figure 9: X-trajectory calculated by the simulation and compared to the average x-trajectory calculated using the equations in this note. Circular polarization right hand magnetic field mode.

Neglecting the curvature term, the inverse focal length is given by

$$\left\langle \frac{dx_L}{dz} \right\rangle_{z=L} = -x_{0L}\frac{1}{f_x} \tag{245}$$

$$\frac{1}{f_x} = \frac{q^2 B_0^2 L}{2\gamma^2 m^2 v_{z0}^2}[\frac{k_r^4}{k_u^4} + \frac{1}{2}\frac{k_s^4}{k_u^4} - \frac{1}{2}\frac{k_r^2 k_s^2}{k_u^4} - \frac{1}{2}\cos(k_u\Delta)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})] \tag{246}$$

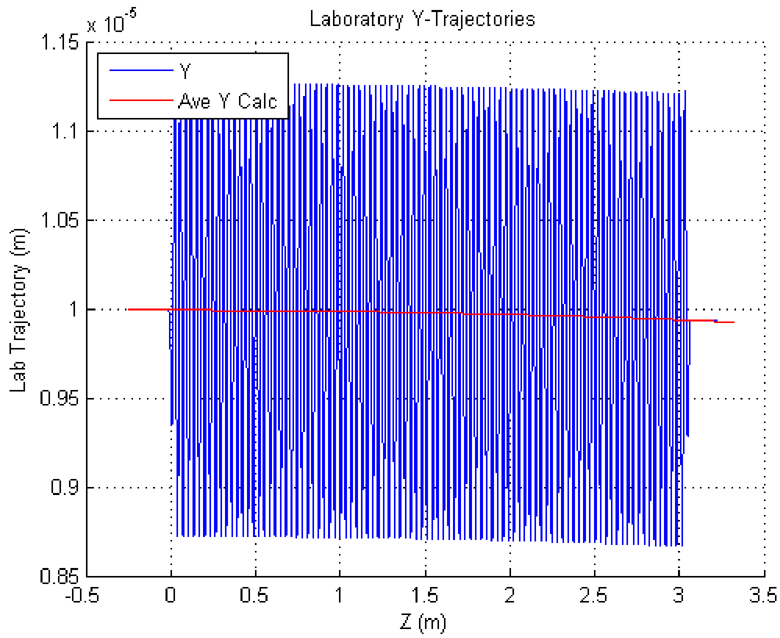

Figure 10: Y-trajectory calculated by the simulation and compared to the average y-trajectory calculated using the equations in this note. Circular polarization right hand magnetic field mode.

$$\left\langle \frac{dy_L}{dz} \right\rangle_{z=L} = -y_{0L}\frac{1}{f_y} \tag{247}$$

$$\frac{1}{f_y} = \frac{q^2 B_0^2 L}{2\gamma^2 m^2 v_{z0}^2}[\frac{k_r^4}{k_u^4} + \frac{1}{2}\frac{k_s^4}{k_u^4} - \frac{1}{2}\frac{k_r^2 k_s^2}{k_u^4} - \frac{1}{2}\cos(k_u\Delta)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})] \tag{248}$$

The inverse focal length is plotted as a function of Δ in figure 11.

The curvature term in the exit slope is independent of the entrance position. This term is

$$\left\langle \frac{dx_L}{dz} \right\rangle = -\frac{q^2 B_0^2 L}{2\sqrt{2}\gamma^2 m^2 v_{z0}^2}\sin(k_u\Delta)\frac{k_r k_s^2}{k_u^4} \tag{249}$$

$$\left\langle \frac{dy_L}{dz} \right\rangle = 0 \tag{250}$$

The curvature term is plotted as a function of Δ in figure 12.

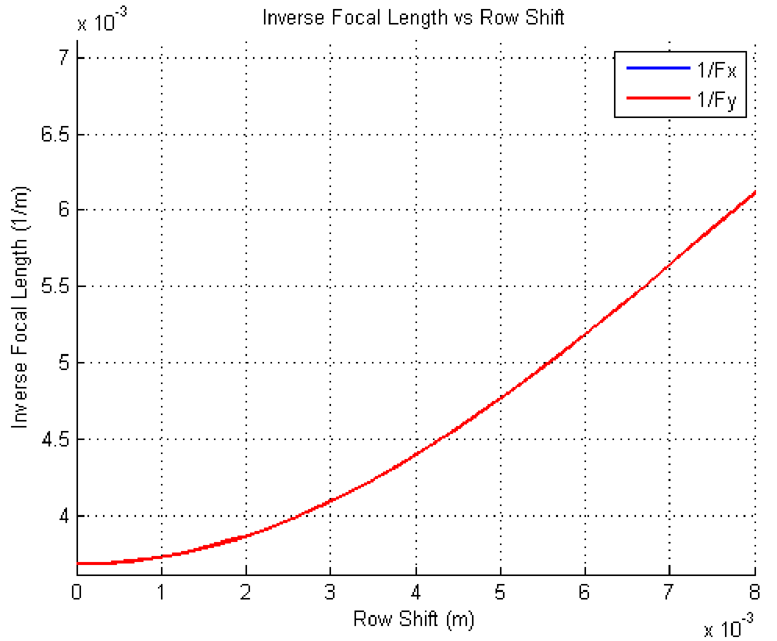

Figure 11: Inverse focal length as a function of Δ in circular right hand magnetic field mode.

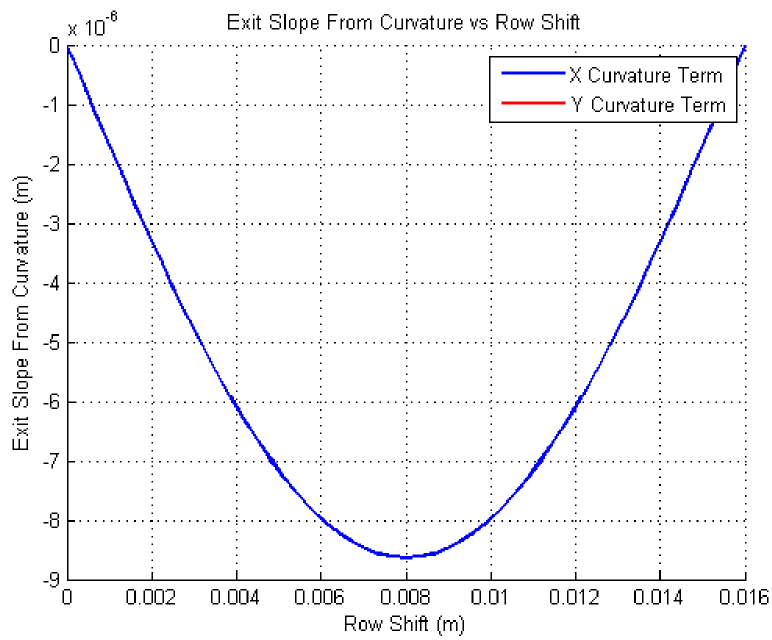

Figure 12: The exit slope has a contribution independent of entrance position. Circular polarization right hand magnetic field mode.

6.8 Circular Polarization Left Hand Magnetic Field

In the circular polarization right hand magnetic field mode,

$$\delta = \frac{\pi}{2} \tag{251}$$

In this case

$$\begin{aligned}\left\langle \frac{dx_L}{dz} \right\rangle &= \frac{q^2 B_0^2}{2\gamma^2 m^2 k_u v_{z0}^2} z\{ \frac{1}{\sqrt{2}} \sin(k_u \Delta) \frac{k_r k_s^2}{k_u^3} \\ &-k_u x_{0L}[\frac{k_r^4}{k_u^4} + \frac{1}{2}\frac{k_s^4}{k_u^4} - \frac{1}{2}\frac{k_r^2 k_s^2}{k_u^4} - \frac{1}{2}\cos(k_u\Delta)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})]\} \end{aligned} \tag{252}$$

$$\begin{aligned}\left\langle \frac{dy_L}{dz} \right\rangle &= \frac{q^2 B_0^2}{2\gamma^2 m^2 k_u v_{z0}^2} z\{ \\ &-k_u y_{0L}[\frac{k_r^4}{k_u^4} + \frac{1}{2}\frac{k_s^4}{k_u^4} - \frac{1}{2}\frac{k_r^2 k_s^2}{k_u^4} - \frac{1}{2}\cos(k_u\Delta)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})]\} \end{aligned} \tag{253}$$

Using the parameter values given above, the trajectories from the simulation are compared to these formulas in figures 13 and 14. In these figures, the row shift Δ is zero corresponding to the maximum K value. The particle enters the undulator at $x_{0L} = 10$ μm, $y_{0L} = 10$ μm with zero initial slope.

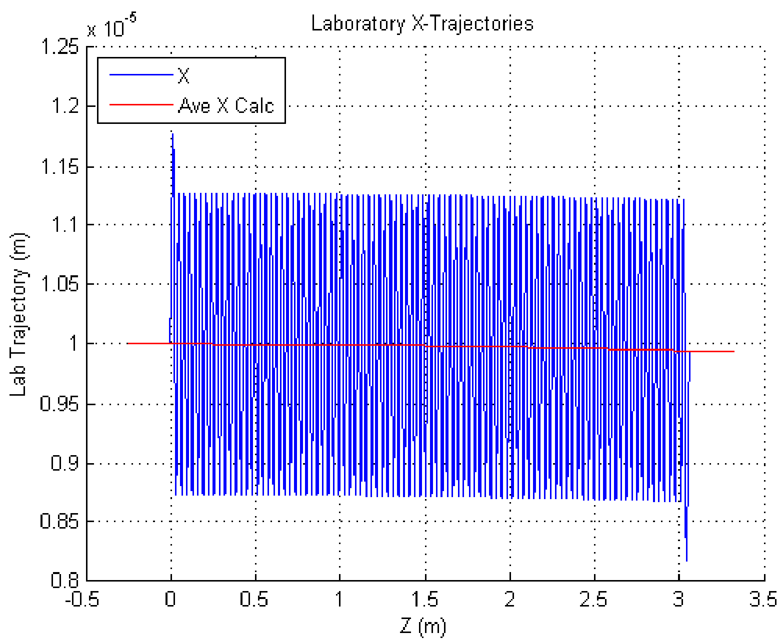

Figure 13: X-trajectory calculated by the simulation and compared to the average x-trajectory calculated using the equations in this note. Circular polarization left hand magnetic field mode.

Neglecting the curvature term, the inverse focal length is given by

$$\left\langle \frac{dx_L}{dz} \right\rangle_{z=L} = -x_{0L}\frac{1}{f_x} \tag{254}$$

$$\frac{1}{f_x} = \frac{q^2 B_0^2 L}{2\gamma^2 m^2 v_{z0}^2}[\frac{k_r^4}{k_u^4} + \frac{1}{2}\frac{k_s^4}{k_u^4} - \frac{1}{2}\frac{k_r^2 k_s^2}{k_u^4} - \frac{1}{2}\cos(k_u\Delta)\frac{k_s^2}{k_u^2}(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2})] \tag{255}$$

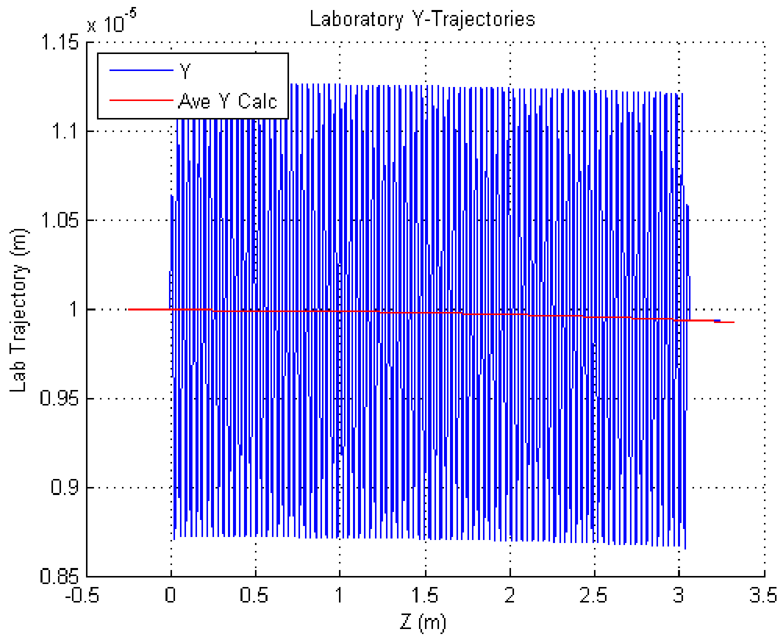

Figure 14: Y-trajectory calculated by the simulation and compared to the average y-trajectory calculated using the equations in this note. Circular polarization left hand magnetic field mode.

$$\left\langle \frac{dy_L}{dz} \right\rangle_{z=L} = -y_{0L}\frac{1}{f_y} \tag{256}$$

$$\frac{1}{f_y} = \frac{q^2 B_0^2 L}{2\gamma^2 m^2 v_{z0}^2}\left[\frac{k_r^4}{k_u^4} + \frac{1}{2}\frac{k_s^4}{k_u^4} - \frac{1}{2}\frac{k_r^2 k_s^2}{k_u^4} - \frac{1}{2}\cos\left(k_u\Delta\right)\frac{k_s^2}{k_u^2}\left(\frac{k_s^2}{k_u^2} + \frac{k_r^2}{k_u^2}\right)\right] \tag{257}$$

The inverse focal length is plotted as a function of Δ in figure 15.

The curvature term in the exit slope is independent of the entrance position. This term is

$$\left\langle \frac{dx_L}{dz} \right\rangle = \frac{q^2 B_0^2 L}{2\sqrt{2}\gamma^2 m^2 v_{z0}^2}\sin\left(k_u\Delta\right)\frac{k_r k_s^2}{k_u^4} \tag{258}$$

$$\left\langle \frac{dy_L}{dz} \right\rangle = 0 \tag{259}$$

The curvature term is plotted as a function of Δ in figure 16.

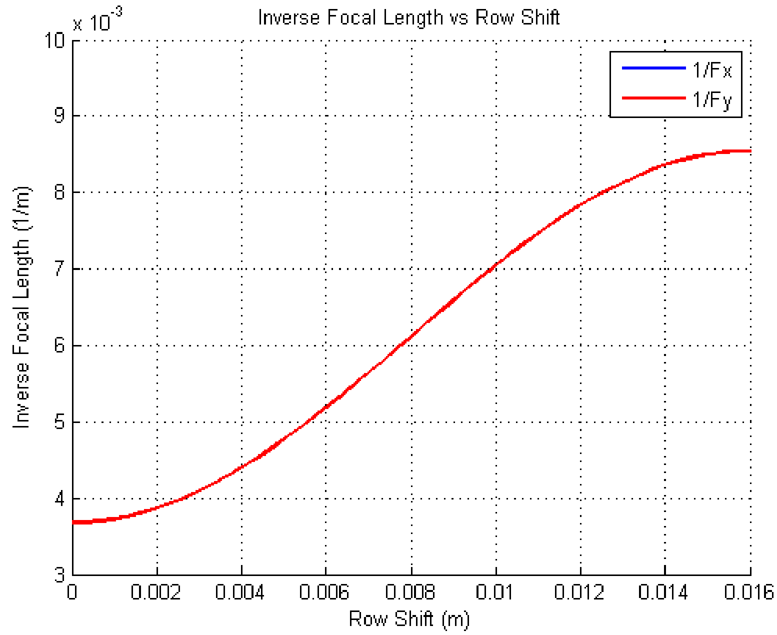

Figure 15: Inverse focal length as a function of Δ in circular left hand magnetic field mode.

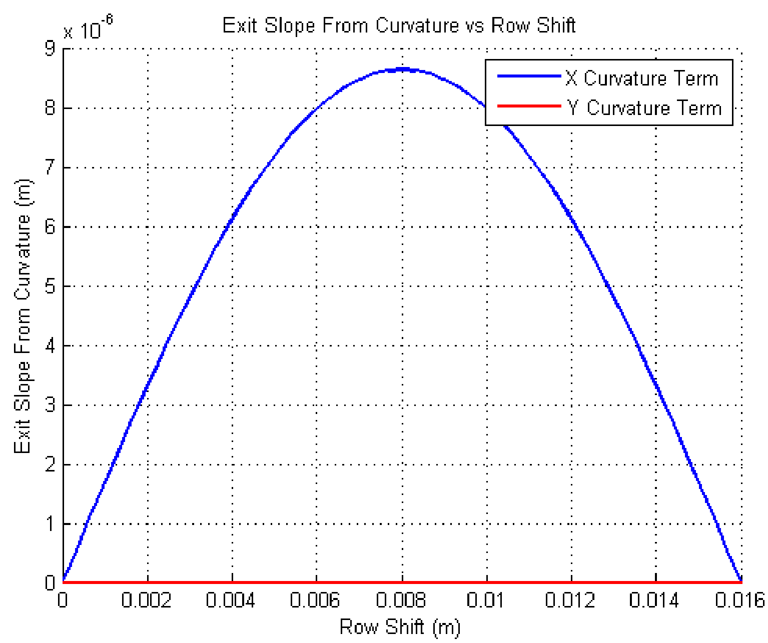

Figure 16: The exit slope has a contribution independent of entrance position. Circular polarization left hand magnetic field mode.

7 Conclusion

The Delta undulator deflects the beam in a complicated way if there is significant field rolloff in the transverse direction away from the center line of a magnet array. If the rolloff is small, then the Delta focuses the beam in both the x_L and y_L directions. If the rolloff is large, the Delta curves the beam, focuses the beam, and deflects the beam in a skew sense (in a general mode), all three simultaneously. If this complicated behavior of the Delta is to be avoided, the block shape must be chosen to minimize field rolloff.

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