Beam Trajectories In The Delta Undulator I

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Abstract

If the field from the permanent magnet quadrants making up the Delta undulator has negligible rolloff away from the center line of the magnets, the undulator behaves as two superposed crossed planar adjustable phase undulators. In this case, the undulator focuses the beam in both transverse directions. The focusing is independent of the mode of the undulator and independent of the K value.

1 Introduction[1]

In this note the magnetic field in the Delta undulator is calculated from the quadrant fields, and the magnetic field is then used to calculate the beam trajectories. An approximation is made that the transverse rolloff of the quadrant fields is negligible in the vicinity of the beam. In this case, the Delta undulator focuses the beam in both the horizontal and vertical directions. The focal length in both directions is the same as for a planar undulator. The focal length is independent of the mode of the undulator and independent of the K value. This result is consistent with treating the undulator as two crossed variable phase planar undulators.

2 Magnetic Field In The Undulator

2.1 The Scalar Potential

In a previous technical note[2] the scalar potential from a single magnet array was derived under the assumption that the transverse rolloff of the potential is negligible in the vicinity of the beam. With this assumption, in the coordinate system shown in figure 1, the scalar potential from a quadrant has the form

$$\phi = \phi_0 \exp\left(-k_u r\right) \cos\left(k_u \left(z - z_0\right)\right) \tag{1}$$

where ϕ_0 is a constant, $k_u = 2\pi/\lambda_u$ where λ_u is the undulator period, z is the coordinate down the undulator, and z_0 gives the quadrant position along z. We work close to the beam axis where the variation of the scalar potential in the s-direction is small and we ignore the s dependence.

We use this form of the potential to calculate the magnetic scalar potential in the undulator by rotating the four quadrants and summing their rotated scalar potentials. The Delta undulator is oriented as shown on the left side of figure 2. In the laboratory, z is along the beam direction, y_L is up, and x_L makes a right handed system. For our calculations, it is more convenient to use the rotated system x, y, z, where x is in the direction from quadrant 3 to quadrant 1, y is in the

[1] Work supported in part by the DOE Contract DE-AC02-76SF00515. This work was performed in support of the LCLS project at SLAC.

[2] Z. Wolf, "A Calculation Of The Fields In The Delta Undulator", LCLS-TN-14-1, January, 2014.

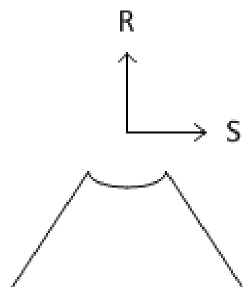

Figure 1: Coordinate system for the scalar potential of a single quadrant.

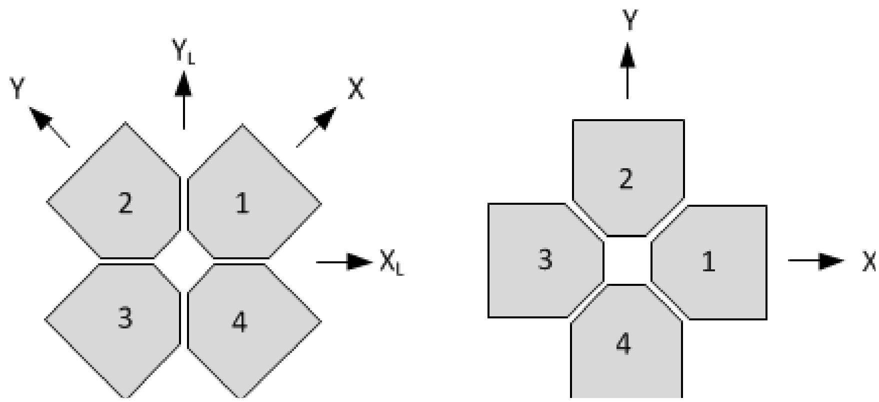

Figure 2: The left side of the figure shows the Delta undulator in its configuration in the tunnel where y_L is up, z is in the beam direction, and x_L makes a right handed system. For our calculations, we use the rotated coordinate system on the right, where x is along the line pointing from quadrant 3 to quadrant 1, y is along the line pointing from quadrant 4 to quadrant 2, and z makes a right handed system.

direction from quadrant 4 to quadrant 2, and z is in the beam direction. Using equation 1, the scalar potential for each of the quadrants in the x, y, z system is

$$\phi_1(x,y,z) = \phi_{0Q} \exp\left(k_u x\right) \cos\left(k_u \left(z - z_{01}\right)\right) \tag{2}$$
$$\phi_2(x,y,z) = \phi_{0Q} \exp\left(k_u y\right) \cos\left(k_u \left(z - z_{02}\right)\right) \tag{3}$$
$$\phi_3(x,y,z) = -\phi_{0Q} \exp\left(-k_u x\right) \cos\left(k_u \left(z - z_{03}\right)\right) \tag{4}$$
$$\phi_4(x,y,z) = -\phi_{0Q} \exp\left(-k_u y\right) \cos\left(k_u \left(z - z_{04}\right)\right) \tag{5}$$

where z_{0i} is the longitudinal shift of quadrant i, and ϕ_{0Q} is the amplitude of the scalar potential of all the identical quadrants on the axis of the undulator where $x = 0$ and $y = 0$. Quadrants 3 and 4 are loaded with opposite polarity magnets as quadrants 1 and 2 in order to make a vertical field planar undulator in the laboratory frame when all the rows are aligned. This accounts for the minus signs, $-\phi_{0Q}$, in the potentials for quadrants 3 and 4.

The scalar potential for the undulator is the sum of the scalar potentials for the quadrants. Quadrants 1 and 3 both depend on x, and quadrants 2 and 4 both depend on y. We first add the scalar potentials for quadrants 1 and 3, and then add the scalar potentials for quadrants 2 and 4, and then add the sums to get the scalar potential for the whole undulator. We will interpret this as forming the entire undulator from two crossed planar adjustable phase undulators.

The scalar potential for the combination of quadrants 1 and 3 is given by

$$\phi_{13} = \phi_{0Q} \exp(k_u x) \cos(k_u (z - z_{01})) - \phi_{0Q} \exp(-k_u x) \cos(k_u (z - z_{03})) \tag{6}$$

Let

$$z_{01} = Z_{13} + \frac{\Delta_{13}}{2} \tag{7}$$

$$z_{03} = Z_{13} - \frac{\Delta_{13}}{2} \tag{8}$$

where

$$Z_{13} = \frac{z_{01} + z_{03}}{2} \tag{9}$$

is the average z-position of the quadrants, and

$$\Delta_{13} = z_{01} - z_{03} \tag{10}$$

is the z-shift between the quadrants. With these definitions, the scalar potential for the pair of quadrants becomes

$$\begin{aligned} \phi_{13} = {} & 2\phi_{0Q} \sinh(k_u x) \cos\left(k_u \frac{\Delta_{13}}{2}\right) \cos(k_u (z - Z_{13})) \\ & + 2\phi_{0Q} \cosh(k_u x) \sin\left(k_u \frac{\Delta_{13}}{2}\right) \sin(k_u (z - Z_{13})) \end{aligned} \tag{11}$$

This is the scalar potential for a planar adjustable phase undulator[3]. The full range of amplitudes and phases are covered if the range of Δ_{13} includes $\left[-\frac{\lambda_u}{2}, \frac{\lambda_u}{2}\right]$ and the range of Z_{13} includes $\left[-\frac{\lambda_u}{2}, \frac{\lambda_u}{2}\right]$.

Similarly, the scalar potential for the combination of quadrants 2 and 4 is given by

$$\phi_{24} = \phi_{0Q} \exp(k_u y) \cos(k_u (z - z_{02})) - \phi_{0Q} \exp(-k_u y) \cos(k_u (z - z_{04})) \tag{12}$$

Let

$$z_{02} = Z_{24} + \frac{\Delta_{24}}{2} \tag{13}$$

$$z_{04} = Z_{24} - \frac{\Delta_{24}}{2} \tag{14}$$

where

$$Z_{24} = \frac{z_{02} + z_{04}}{2} \tag{15}$$

is the average z-position of the quadrants, and

$$\Delta_{24} = z_{02} - z_{04} \tag{16}$$

is the z-shift between the quadrants.

With these definitions, the scalar potential for the pair of quadrants becomes

$$\begin{aligned} \phi_{24} = {} & 2\phi_{0Q} \sinh(k_u y) \cos\left(k_u \frac{\Delta_{24}}{2}\right) \cos(k_u (z - Z_{24})) \\ & + 2\phi_{0Q} \cosh(k_u y) \sin\left(k_u \frac{\Delta_{24}}{2}\right) \sin(k_u (z - Z_{24})) \end{aligned} \tag{17}$$

[3] Z. Wolf, "Variable Phase PPM Undulator Study", LCLS-TN-11-1, May, 2011.

This is again the potential for a planar adjustable phase undulator.

The scalar potential for the undulator is the sum of the scalar potentials for the quadrant pairs.

$$\phi = \phi_{13} + \phi_{24} \tag{18}$$

Performing the sum, we find

$$\begin{aligned}\phi &= 2\phi_{0Q}\sinh\left(k_u x\right)\cos\left(k_u\frac{\Delta_{13}}{2}\right)\cos\left(k_u\left(z-Z_{13}\right)\right)\\ &+2\phi_{0Q}\cosh\left(k_u x\right)\sin\left(k_u\frac{\Delta_{13}}{2}\right)\sin\left(k_u\left(z-Z_{13}\right)\right)\\ &+2\phi_{0Q}\sinh\left(k_u y\right)\cos\left(k_u\frac{\Delta_{24}}{2}\right)\cos\left(k_u\left(z-Z_{24}\right)\right)\\ &+2\phi_{0Q}\cosh\left(k_u y\right)\sin\left(k_u\frac{\Delta_{24}}{2}\right)\sin\left(k_u\left(z-Z_{24}\right)\right)\end{aligned} \tag{19}$$

By putting in the various values for Z_{13}, Z_{24}, Δ_{13}, and Δ_{24}, we get the scalar potential for the various undulator modes at different K values.

2.2 Magnetic Field In The Undulator Coordinate System

The magnetic field in the undulator is given by $B = \nabla\phi$. Taking the gradient we find

$$\begin{aligned}B_x\left(x,y,z\right) &= 2\phi_{0Q}k_u[\cos\left(k_u\frac{\Delta_{13}}{2}\right)\cosh\left(k_u x\right)\cos\left(k_u\left(z-Z_{13}\right)\right)\\ &+\sin\left(k_u\frac{\Delta_{13}}{2}\right)\sinh\left(k_u x\right)\sin\left(k_u\left(z-Z_{13}\right)\right)]\end{aligned} \tag{20}$$

$$\begin{aligned}B_y\left(x,y,z\right) &= 2\phi_{0Q}k_u[\cos\left(k_u\frac{\Delta_{24}}{2}\right)\cosh\left(k_u y\right)\cos\left(k_u\left(z-Z_{24}\right)\right)\\ &+\sin\left(k_u\frac{\Delta_{24}}{2}\right)\sinh\left(k_u y\right)\sin\left(k_u\left(z-Z_{24}\right)\right)]\end{aligned} \tag{21}$$

$$\begin{aligned}B_z\left(x,y,z\right) &= 2\phi_{0Q}k_u[-\cos\left(k_u\frac{\Delta_{13}}{2}\right)\sinh\left(k_u x\right)\sin\left(k_u\left(z-Z_{13}\right)\right)\\ &+\sin\left(k_u\frac{\Delta_{13}}{2}\right)\cosh\left(k_u x\right)\cos\left(k_u\left(z-Z_{13}\right)\right)\\ &-\cos\left(k_u\frac{\Delta_{24}}{2}\right)\sinh\left(k_u y\right)\sin\left(k_u\left(z-Z_{24}\right)\right)\\ &+\sin\left(k_u\frac{\Delta_{24}}{2}\right)\cosh\left(k_u y\right)\cos\left(k_u\left(z-Z_{24}\right)\right)]\end{aligned} \tag{22}$$

In order to simplify these formulas further, let

$$B_0 = 2\phi_{0Q}k_u \tag{23}$$

$$B_{xc} = B_0\cos\left(k_u\frac{\Delta_{13}}{2}\right) \tag{24}$$

$$B_{xs} = B_0\sin\left(k_u\frac{\Delta_{13}}{2}\right) \tag{25}$$

$$B_{yc} = B_0\cos\left(k_u\frac{\Delta_{24}}{2}\right) \tag{26}$$

$$B_{ys} = B_0\sin\left(k_u\frac{\Delta_{24}}{2}\right) \tag{27}$$

With these substitutions, the fields become

$$\begin{aligned}
B_x(x,y,z) &= B_{xc}\cosh(k_u x)\cos(k_u(z-Z_{13})) + B_{xs}\sinh(k_u x)\sin(k_u(z-Z_{13})) \quad (28)\\
B_y(x,y,z) &= B_{yc}\cosh(k_u y)\cos(k_u(z-Z_{24})) + B_{ys}\sinh(k_u y)\sin(k_u(z-Z_{24})) \quad (29)\\
B_z(x,y,z) &= -B_{xc}\sinh(k_u x)\sin(k_u(z-Z_{13})) + B_{xs}\cosh(k_u x)\cos(k_u(z-Z_{13}))\\
&\quad -B_{yc}\sinh(k_u y)\sin(k_u(z-Z_{24})) + B_{ys}\cosh(k_u y)\cos(k_u(z-Z_{24})) \quad (30)
\end{aligned}$$

We can change the origin of z such that

$$z = z' + Z_{13} \quad (31)$$

If we let

$$\delta = k_u(Z_{13} - Z_{24}) \quad (32)$$

and drop the prime, we write the fields as a function of position as

$$\begin{aligned}
B_x &= B_{xc}\cosh(k_u x)\cos(k_u z) + B_{xs}\sinh(k_u x)\sin(k_u z) \quad (33)\\
B_y &= B_{yc}\cosh(k_u y)\cos(k_u z+\delta) + B_{ys}\sinh(k_u y)\sin(k_u z+\delta) \quad (34)\\
B_z &= -B_{xc}\sinh(k_u x)\sin(k_u z) + B_{xs}\cosh(k_u x)\cos(k_u z)\\
&\quad -B_{yc}\sinh(k_u y)\sin(k_u z+\delta) + B_{ys}\cosh(k_u y)\cos(k_u z+\delta) \quad (35)
\end{aligned}$$

2.3 Magnetic Field In The Laboratory Frame

The scalar potential does not change value going to the laboratory frame, only the coordinates specifying a point change. Using

$$\begin{aligned}
x &= \frac{1}{\sqrt{2}}(x_L + y_L) \quad (36)\\
y &= \frac{1}{\sqrt{2}}(y_L - x_L) \quad (37)
\end{aligned}$$

in equation 19, we find

$$\begin{aligned}
\phi &= 2\phi_{0Q}\sinh\left(k_u\frac{1}{\sqrt{2}}(x_L+y_L)\right)\cos\left(k_u\frac{\Delta_{13}}{2}\right)\cos(k_u(z-Z_{13}))\\
&\quad +2\phi_{0Q}\cosh\left(k_u\frac{1}{\sqrt{2}}(x_L+y_L)\right)\sin\left(k_u\frac{\Delta_{13}}{2}\right)\sin(k_u(z-Z_{13}))\\
&\quad +2\phi_{0Q}\sinh\left(k_u\frac{1}{\sqrt{2}}(y_L-x_L)\right)\cos\left(k_u\frac{\Delta_{24}}{2}\right)\cos(k_u(z-Z_{24}))\\
&\quad +2\phi_{0Q}\cosh\left(k_u\frac{1}{\sqrt{2}}(y_L-x_L)\right)\sin\left(k_u\frac{\Delta_{24}}{2}\right)\sin(k_u(z-Z_{24})) \quad (38)
\end{aligned}$$

The magnetic fields are

$$\begin{aligned}
B_{xL} &= \frac{B_{xc}}{\sqrt{2}}\cosh\left(k_u\frac{1}{\sqrt{2}}(x_L+y_L)\right)\cos(k_u z)\\
&\quad +\frac{B_{xs}}{\sqrt{2}}\sinh\left(k_u\frac{1}{\sqrt{2}}(x_L+y_L)\right)\sin(k_u z)\\
&\quad -\frac{B_{yc}}{\sqrt{2}}\cosh\left(k_u\frac{1}{\sqrt{2}}(y_L-x_L)\right)\cos(k_u z+\delta)\\
&\quad -\frac{B_{ys}}{\sqrt{2}}\sinh\left(k_u\frac{1}{\sqrt{2}}(y_L-x_L)\right)\sin(k_u z+\delta) \quad (39)
\end{aligned}$$

$$
\begin{aligned}
B_{yL} &= \frac{B_{xc}}{\sqrt{2}}\cosh\left(k_u\frac{1}{\sqrt{2}}\left(x_L+y_L\right)\right)\cos\left(k_u z\right) \\
&+\frac{B_{xs}}{\sqrt{2}}\sinh\left(k_u\frac{1}{\sqrt{2}}\left(x_L+y_L\right)\right)\sin\left(k_u z\right) \\
&+\frac{B_{yc}}{\sqrt{2}}\cosh\left(k_u\frac{1}{\sqrt{2}}\left(y_L-x_L\right)\right)\cos\left(k_u z+\delta\right) \\
&+\frac{B_{ys}}{\sqrt{2}}\sinh\left(k_u\frac{1}{\sqrt{2}}\left(y_L-x_L\right)\right)\sin\left(k_u z+\delta\right) \qquad (40)
\end{aligned}
$$

$$
\begin{aligned}
B_{zL} &= -B_{xc}\sinh\left(k_u\frac{1}{\sqrt{2}}\left(x_L+y_L\right)\right)\sin\left(k_u z\right) \\
&+B_{xs}\cosh\left(k_u\frac{1}{\sqrt{2}}\left(x_L+y_L\right)\right)\cos\left(k_u z\right) \\
&-B_{yc}\sinh\left(k_u\frac{1}{\sqrt{2}}\left(y_L-x_L\right)\right)\sin\left(k_u z+\delta\right) \\
&+B_{ys}\cosh\left(k_u\frac{1}{\sqrt{2}}\left(y_L-x_L\right)\right)\cos\left(k_u z+\delta\right) \qquad (41)
\end{aligned}
$$

These fields are significantly more complicated than the fields in the undulator coordinate system. They are useful for simulations, but for analytic calculations, we return to the undulator coordinate system.

2.4 Quadrant Positions

For given magnetic field parameters, we can determine the quadrant positions. There are three adjustable magnetic field parameters, Δ_{13}, Δ_{24}, and δ, so we need one additional constraint equation in order to determine the four quadrant positions. We choose the extra constraint to be that the average z-position of the quadrants is zero.

$$
z_{01}+z_{02}+z_{03}+z_{04}=0 \qquad (42)
$$

The other three equations are

$$
\begin{aligned}
z_{01}-z_{03} &= \Delta_{13} \qquad (43)\\
z_{02}-z_{04} &= \Delta_{24} \qquad (44)\\
z_{01}-z_{02}+z_{03}-z_{04} &= \frac{2}{k_u}\delta \qquad (45)
\end{aligned}
$$

The solution for the row positions in terms of the field specifying parameters is

$$
\begin{aligned}
z_{01} &= \frac{1}{2}\left(\frac{\delta}{k_u}+\Delta_{13}\right) \qquad (46)\\
z_{02} &= \frac{1}{2}\left(-\frac{\delta}{k_u}+\Delta_{24}\right) \qquad (47)\\
z_{03} &= \frac{1}{2}\left(\frac{\delta}{k_u}-\Delta_{13}\right) \qquad (48)\\
z_{04} &= \frac{1}{2}\left(-\frac{\delta}{k_u}-\Delta_{24}\right) \qquad (49)
\end{aligned}
$$

3 Beam Trajectory And Focusing

3.1 Equations Of Motion

The trajectory of a charged particle beam in the undulator is determined by the Lorentz force law

$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{v} \times \mathbf{B}\right) \tag{50}$$

where $\mathbf{p} = \gamma m \mathbf{v}$, $\gamma = \left(1-\beta^2\right)^{-1}$, and m is the particle rest mass. The energy of the particle is constant in the magnetic field of the undulator, except for radiation losses which we neglect. With constant energy, γ is constant. The Lorentz force law becomes

$$\dot{\mathbf{v}} = \frac{q}{\gamma m}\left(\mathbf{v} \times \mathbf{B}\right) \tag{51}$$

With the substitution $d/dt = v_z d/dz$, the changes in the individual velocity components with z are given by

$$\frac{dv_x}{dz} = \frac{q}{\gamma m v_z}\left(v_y B_z - v_z B_y\right) \tag{52}$$

$$\frac{dv_y}{dz} = \frac{q}{\gamma m v_z}\left(v_z B_x - v_x B_z\right) \tag{53}$$

$$\frac{dv_z}{dz} = \frac{q}{\gamma m v_z}\left(v_x B_y - v_y B_x\right) \tag{54}$$

3.2 Iterative Solution

We solve these equations iteratively by expanding in powers of a small parameter. Let

$$\epsilon = \frac{qB_0}{\gamma m k_u c} \tag{55}$$

be the small dimensionless expansion parameter. In terms of this parameter, the equations of motion are

$$\frac{dv_x}{dz} = \epsilon\frac{k_u c}{B_0 v_z}\left(v_y B_z - v_z B_y\right) \tag{56}$$

$$\frac{dv_y}{dz} = \epsilon\frac{k_u c}{B_0 v_z}\left(v_z B_x - v_x B_z\right) \tag{57}$$

$$\frac{dv_z}{dz} = \epsilon\frac{k_u c}{B_0 v_z}\left(v_x B_y - v_y B_x\right) \tag{58}$$

At this point we could divide all velocities by c, divide all positions by $1/k_u$, and divide all magnetic fields by B_0 in order to eliminate the factor $\frac{k_u c}{B_0 v_z}$. We think of small quantities in terms of these dimensionless variables. In the interest of clarity, however, we keep the factor and continue to use mks units, so there is no confusion about the meaning of a quantity.

Expanding the particle position in terms of the expansion parameter, we have

$$x = x_{(0)} + \epsilon x_{(1)} + \epsilon^2 x_{(2)} + \cdots \tag{59}$$

$$y = y_{(0)} + \epsilon y_{(1)} + \epsilon^2 y_{(2)} + \cdots \tag{60}$$

$$z = z \tag{61}$$

where the last line indicates that the z-position of the particle is the independent variable. The numbers in parenthesis indicate the order of the expansion. The velocities are

$$v_x = v_{x(0)} + \epsilon v_{x(1)} + \epsilon^2 v_{x(2)} + \cdots \tag{62}$$
$$v_y = v_{y(0)} + \epsilon v_{y(1)} + \epsilon^2 v_{y(2)} + \cdots \tag{63}$$
$$v_z = v_{z(0)} + \epsilon v_{z(1)} + \epsilon^2 v_{z(2)} + \cdots \tag{64}$$

The magnetic fields are given by

$$B_x = B_x|_0 + \epsilon\partial_x B_x|_0 x_{(1)} + \epsilon\partial_y B_x|_0 y_{(1)} + \cdots \tag{65}$$
$$B_y = B_y|_0 + \epsilon\partial_x B_y|_0 x_{(1)} + \epsilon\partial_y B_y|_0 y_{(1)} + \cdots \tag{66}$$
$$B_z = B_z|_0 + \epsilon\partial_x B_z|_0 x_{(1)} + \epsilon\partial_y B_z|_0 y_{(1)} + \cdots \tag{67}$$

where

$$B_i|_0 = B_i(x_{(0)}, y_{(0)}, z) \tag{68}$$
$$\partial_j B_i|_0 = \frac{\partial B_i}{\partial x_j}(x_{(0)}, y_{(0)}, z) \tag{69}$$

where $i = x, y, z$, $j = x, y$.

The transverse positions are found from the velocities as follows:

$$v_x = \frac{dx}{dz}\frac{dz}{dt} \tag{70}$$
$$\begin{aligned} x(z) &= x_0 + \int_0^z v_x(z')\frac{1}{v_z(z')}dz' \\ &= x_0 + \int_0^z \left(v_{x(0)} + \epsilon v_{x(1)} + \cdots\right)\frac{1}{v_{z(0)}}\left(1 - \epsilon\frac{v_{z(1)}}{v_{z(0)}} + \cdots\right)dz' \\ &= x_0 + \frac{v_{x(0)}}{v_{z(0)}}z + \epsilon\int_0^z \left(\frac{v_{x(1)}}{v_{z(0)}} - \frac{v_{x(0)}v_{z(1)}}{v_{z(0)}^2}\right)dz' + \cdots \end{aligned} \tag{71}$$

In these equations x_0 is the constant initial position where the beam enters the undulator. Similarly

$$y(z) = y_0 + \frac{v_{y(0)}}{v_{z(0)}}z + \epsilon\int_0^z \left(\frac{v_{y(1)}}{v_{z(0)}} - \frac{v_{y(0)}v_{z(1)}}{v_{z(0)}^2}\right)dz' + \cdots \tag{72}$$

We see that

$$x_{(0)} = x_0 + \frac{v_{x(0)}}{v_{z(0)}}z \tag{73}$$
$$y_{(0)} = y_0 + \frac{v_{y(0)}}{v_{z(0)}}z \tag{74}$$

and

$$x_{(1)} = \int_0^z \left(\frac{v_{x(1)}}{v_{z(0)}} - \frac{v_{x(0)}v_{z(1)}}{v_{z(0)}^2}\right)dz' \tag{75}$$
$$y_{(1)} = \int_0^z \left(\frac{v_{y(1)}}{v_{z(0)}} - \frac{v_{y(0)}v_{z(1)}}{v_{z(0)}^2}\right)dz' \tag{76}$$

With the expansions in the small parameter ϵ, the equations of motion become

$$\begin{aligned}
& \frac{dv_{x(0)}}{dz} + \epsilon\frac{dv_{x(1)}}{dz} + \epsilon^2\frac{dv_{x(2)}}{dz} + \cdots \\
= & \epsilon\frac{k_u c}{B_0 v_{z(0)}}\left[1 - \epsilon\frac{v_{z(1)}}{v_{z(0)}} + \cdots\right] \\
& \times\{(v_{y(0)} + \epsilon v_{y(1)} + \cdots)(B_z|_0 + \epsilon\partial_x B_z|_0 x_{(1)} + \epsilon\partial_y B_z|_0 y_{(1)} + \cdots) \\
& -(v_{z(0)} + \epsilon v_{z(1)} + \cdots)(B_y|_0 + \epsilon\partial_x B_y|_0 x_{(1)} + \epsilon\partial_y B_y|_0 y_{(1)} + \cdots)\}
\end{aligned} \tag{77}$$

$$\begin{aligned}
& \frac{dv_{y(0)}}{dz} + \epsilon\frac{dv_{y(1)}}{dz} + \epsilon^2\frac{dv_{y(2)}}{dz} + \cdots \\
= & \epsilon\frac{k_u c}{B_0 v_{z(0)}}\left[1 - \epsilon\frac{v_{z(1)}}{v_{z(0)}} + \cdots\right] \\
& \times\{(v_{z(0)} + \epsilon v_{z(1)} + \cdots)(B_x|_0 + \epsilon\partial_x B_x|_0 x_{(1)} + \epsilon\partial_y B_x|_0 y_{(1)} + \cdots) \\
& -(v_{x(0)} + \epsilon v_{x(1)} + \cdots)(B_z|_0 + \epsilon\partial_x B_z|_0 x_{(1)} + \epsilon\partial_y B_z|_0 y_{(1)} + \cdots)\}
\end{aligned} \tag{78}$$

$$\begin{aligned}
& \frac{dv_{z(0)}}{dz} + \epsilon\frac{dv_{z(1)}}{dz} + \epsilon^2\frac{dv_{z(2)}}{dz} + \cdots \\
= & \epsilon\frac{k_u c}{B_0 v_{z(0)}}\left[1 - \epsilon\frac{v_{z(1)}}{v_{z(0)}} + \cdots\right] \\
& \times\{(v_{x(0)} + \epsilon v_{x(1)} + \cdots)(B_y|_0 + \epsilon\partial_x B_y|_0 x_{(1)} + \epsilon\partial_y B_y|_0 y_{(1)} + \cdots) \\
& -(v_{y(0)} + \epsilon v_{y(1)} + \cdots)(B_x|_0 + \epsilon\partial_x B_x|_0 x_{(1)} + \epsilon\partial_y B_x|_0 y_{(1)} + \cdots)\}
\end{aligned} \tag{79}$$

We now proceed to solve these equations for zeroth, first, and second orders.

3.2.1 Zeroth Order

To zeroth order in ϵ, the equations of motion are

$$\frac{dv_{x(0)}}{dz} = 0 \tag{80}$$

$$\frac{dv_{y(0)}}{dz} = 0 \tag{81}$$

$$\frac{dv_{z(0)}}{dz} = 0 \tag{82}$$

The solutions can be written by inspection

$$v_{x(0)} = v_{x0} \tag{83}$$

$$v_{y(0)} = v_{y0} \tag{84}$$

$$v_{z(0)} = v_{z0} \tag{85}$$

where the v_{x0}, v_{y0}, v_{z0} are the constant initial velocities. The corresponding zeroth order transverse positions are

$$x_{(0)} = x_0 + \frac{v_{x0}}{v_{z0}}z \tag{86}$$

$$y_{(0)} = y_0 + \frac{v_{y0}}{v_{z0}}z \tag{87}$$

where the x_0 and y_0 have been previously defined to be the initial positions.

At this point, we make a simplifying assumption. We assume that all particles have zero initial transverse velocity. In this case

$$\begin{aligned} v_{x(0)} &= 0 \quad (88) \\ v_{y(0)} &= 0 \quad (89) \\ v_{z(0)} &= v_{z0} \quad (90) \end{aligned}$$

and

$$\begin{aligned} x_{(0)} &= x_0 \quad (91) \\ y_{(0)} &= y_0 \quad (92) \end{aligned}$$

3.2.2 First Order

To first order in ϵ, the equations of motion are

$$\begin{aligned} \frac{dv_{x(1)}}{dz} &= \frac{k_u c}{B_0 v_{z(0)}}\{v_{y(0)} B_z|_0 - v_{z(0)} B_y|_0\} \quad (93) \\ \frac{dv_{y(1)}}{dz} &= \frac{k_u c}{B_0 v_{z(0)}}\{v_{z(0)} B_x|_0 - v_{x(0)} B_z|_0\} \quad (94) \\ \frac{dv_{z(1)}}{dz} &= \frac{k_u c}{B_0 v_{z(0)}}\{v_{x(0)} B_y|_0 - v_{y(0)} B_x|_0\} \quad (95) \end{aligned}$$

With zero initial transverse velocity, these equations become

$$\begin{aligned} \frac{dv_{x(1)}}{dz} &= -\frac{k_u c}{B_0} B_y|_0 \quad (96) \\ \frac{dv_{y(1)}}{dz} &= \frac{k_u c}{B_0} B_x|_0 \quad (97) \\ \frac{dv_{z(1)}}{dz} &= 0 \quad (98) \end{aligned}$$

Inserting the expressions for the fields in the undulator coordinate system and integrating, we find the velocity components to first order:

$$\begin{aligned} v_{x(1)} &= -\frac{c}{B_0}\left[B_{yc}\cosh(k_u y_0)\sin(k_u z+\delta) - B_{ys}\sinh(k_u y_0)\cos(k_u z+\delta)\right] \quad (99) \\ v_{y(1)} &= \frac{c}{B_0}\left[B_{xc}\cosh(k_u x_0)\sin(k_u z) - B_{xs}\sinh(k_u x_0)\cos(k_u z)\right] \quad (100) \\ v_{z(1)} &= 0 \quad (101) \end{aligned}$$

We use these first order velocities to find the first order corrections to the transverse position of a particle. Using equation 75 with $v_{x(0)} = 0$, we find

$$\begin{aligned} x_{(1)} &= \int_0^z \left(\frac{v_{x(1)}}{v_{z(0)}}\right) dz' \\ &= \frac{c}{B_0 v_{z(0)} k_u}\left[B_{yc}\cosh(k_u y_0)\cos(k_u z+\delta) + B_{ys}\sinh(k_u y_0)\sin(k_u z+\delta)\right] \quad (102) \end{aligned}$$

Similarly

$$\begin{aligned} y_{(1)} &= \int_0^z \left(\frac{v_{y(1)}}{v_{z(0)}}\right) dz' \\ &= -\frac{c}{B_0 v_{z(0)} k_u}\left[B_{xc}\cosh(k_u x_0)\cos(k_u z) + B_{xs}\sinh(k_u x_0)\sin(k_u z)\right] \quad (103) \end{aligned}$$

3.2.3 Second Order

To second order in ϵ, the equations of motion are

$$\begin{aligned}\frac{dv_{x(2)}}{dz} &= \frac{k_u c}{B_0 v_{z(0)}}\{v_{y(1)}B_z|_0 + v_{y(0)}[\partial_x B_z|_0 x_{(1)} + \partial_y B_z|_0 y_{(1)}] \\ &\quad - v_{z(1)}B_y|_0 - v_{z(0)}[\partial_x B_y|_0 x_{(1)} + \partial_y B_y|_0 y_{(1)}] \\ &\quad - \frac{v_{z(1)}}{v_{z(0)}} v_{y(0)} B_z|_0 + \frac{v_{z(1)}}{v_{z(0)}} v_{z(0)} B_y|_0\} \end{aligned} \tag{104}$$

$$\begin{aligned}\frac{dv_{y(2)}}{dz} &= \frac{k_u c}{B_0 v_{z(0)}}\{v_{z(1)}B_x|_0 + v_{z(0)}[\partial_x B_x|_0 x_{(1)} + \partial_y B_x|_0 y_{(1)}] \\ &\quad - v_{x(1)}B_z|_0 - v_{x(0)}[\partial_x B_z|_0 x_{(1)} + \partial_y B_z|_0 y_{(1)}] \\ &\quad - \frac{v_{z(1)}}{v_{z(0)}} v_{z(0)} B_x|_0 + \frac{v_{z(1)}}{v_{z(0)}} v_{x(0)} B_z|_0\} \end{aligned} \tag{105}$$

$$\begin{aligned}\frac{dv_{z(2)}}{dz} &= \frac{k_u c}{B_0 v_{z(0)}}\{v_{x(1)}B_y|_0 + v_{x(0)}[\partial_x B_y|_0 x_{(1)} + \partial_y B_y|_0 y_{(1)}] \\ &\quad - v_{y(1)}B_x|_0 - v_{y(0)}[\partial_x B_x|_0 x_{(1)} + \partial_y B_x|_0 y_{(1)}] \\ &\quad - \frac{v_{z(1)}}{v_{z(0)}} v_{x(0)} B_y|_0 + \frac{v_{z(1)}}{v_{z(0)}} v_{y(0)} B_x|_0\} \end{aligned} \tag{106}$$

Inserting zero zeroth order transverse velocities and zero first order longitudinal velocity, we have

$$\frac{dv_{x(2)}}{dz} = \frac{k_u c}{B_0 v_{z(0)}}\{v_{y(1)}B_z|_0 - v_{z(0)}[\partial_x B_y|_0 x_{(1)} + \partial_y B_y|_0 y_{(1)}]\} \tag{107}$$

$$\frac{dv_{y(2)}}{dz} = \frac{k_u c}{B_0 v_{z(0)}}\{v_{z(0)}[\partial_x B_x|_0 x_{(1)} + \partial_y B_x|_0 y_{(1)}] - v_{x(1)}B_z|_0\} \tag{108}$$

$$\frac{dv_{z(2)}}{dz} = \frac{k_u c}{B_0 v_{z(0)}}\{v_{x(1)}B_y|_0 - v_{y(1)}B_x|_0\} \tag{109}$$

Note that in the undulator frame, B_x depends only on x and B_y depends only on y. This simplifies these equations to the following:

$$\frac{dv_{x(2)}}{dz} = \frac{k_u c}{B_0 v_{z(0)}}\{v_{y(1)}B_z|_0 - v_{z(0)}\partial_y B_y|_0 y_{(1)}\} \tag{110}$$

$$\frac{dv_{y(2)}}{dz} = \frac{k_u c}{B_0 v_{z(0)}}\{v_{z(0)}\partial_x B_x|_0 x_{(1)} - v_{x(1)}B_z|_0\} \tag{111}$$

$$\frac{dv_{z(2)}}{dz} = \frac{k_u c}{B_0 v_{z(0)}}\{v_{x(1)}B_y|_0 - v_{y(1)}B_x|_0\} \tag{112}$$

We must now insert the expressions for the first order velocities, the fields, and the derivatives of the fields.

$$\begin{aligned}\frac{dv_{x(2)}}{dz} &= \frac{k_u c}{B_0 v_{z(0)}}\{\frac{c}{B_0}[B_{xc}\cosh(k_u x_0)\sin(k_u z) - B_{xs}\sinh(k_u x_0)\cos(k_u z)] \\ &\quad \times[-B_{xc}\sinh(k_u x_0)\sin(k_u z) + B_{xs}\cosh(k_u x_0)\cos(k_u z) \\ &\quad - B_{yc}\sinh(k_u y_0)\sin(k_u z+\delta) + B_{ys}\cosh(k_u y_0)\cos(k_u z+\delta)] \\ &\quad - v_{z(0)}[B_{yc}k_u\sinh(k_u y_0)\cos(k_u z+\delta) + B_{ys}k_u\cosh(k_u y_0)\sin(k_u z+\delta)] \\ &\quad \times\left(-\frac{c}{B_0 v_{z(0)} k_u}\right)[B_{xc}\cosh(k_u x_0)\cos(k_u z) + B_{xs}\sinh(k_u x_0)\sin(k_u z)]\} \end{aligned} \tag{113}$$

$$\frac{dv_{y(2)}}{dz} = \frac{k_u c}{B_0 v_{z(0)}} \{ v_{z(0)} [B_{xc} k_u \sinh(k_u x_0) \cos(k_u z) + B_{xs} k_u \cosh(k_u x_0) \sin(k_u z)] \times \left(\frac{c}{B_0 v_{z(0)} k_u} \right) [B_{yc} \cosh(k_u y_0) \cos(k_u z + \delta) + B_{ys} \sinh(k_u y_0) \sin(k_u z + \delta)] - \left(-\frac{c}{B_0} \right) [B_{yc} \cosh(k_u y_0) \sin(k_u z + \delta) - B_{ys} \sinh(k_u y_0) \cos(k_u z + \delta)] \times [-B_{xc} \sinh(k_u x_0) \sin(k_u z) + B_{xs} \cosh(k_u x_0) \cos(k_u z) - B_{yc} \sinh(k_u y_0) \sin(k_u z + \delta) + B_{ys} \cosh(k_u y_0) \cos(k_u z + \delta)] \} \quad (114)$$

$$\frac{dv_{z(2)}}{dz} = \frac{k_u c}{B_0 v_{z(0)}} \{ -\frac{c}{B_0} [B_{yc} \cosh(k_u y_0) \sin(k_u z + \delta) - B_{ys} \sinh(k_u y_0) \cos(k_u z + \delta)] \times [B_{yc} \cosh(k_u y_0) \cos(k_u z + \delta) + B_{ys} \sinh(k_u y_0) \sin(k_u z + \delta)] - \frac{c}{B_0} [B_{xc} \cosh(k_u x_0) \sin(k_u z) - B_{xs} \sinh(k_u x_0) \cos(k_u z)] \times [B_{xc} \cosh(k_u x_0) \cos(k_u z) + B_{xs} \sinh(k_u x_0) \sin(k_u z)] \} \quad (115)$$

These equations consist of products of terms that vary sinusoidally in z. They result in terms that have no z-dependence and terms that are oscillating as $\cos(2k_u z)$ or $\sin(2k_u z)$. In this note we only consider the terms that do not depend on z. These terms lead to average trajectory deviations from a straight line. We denote the z-independent terms with $\langle\rangle$.

$$\left\langle \frac{dv_{x(2)}}{dz} \right\rangle = \frac{k_u c^2}{2 B_0^2 v_{z(0)}} \{ -B_{xc} \cosh(k_u x_0)\, B_{xc} \sinh(k_u x_0) - B_{xc} \cosh(k_u x_0)\, B_{yc} \sinh(k_u y_0) \cos(\delta) - B_{xc} \cosh(k_u x_0)\, B_{ys} \cosh(k_u y_0) \sin(\delta) - B_{xs} \sinh(k_u x_0)\, B_{xs} \cosh(k_u x_0) + B_{xs} \sinh(k_u x_0)\, B_{yc} \sinh(k_u y_0) \sin(\delta) - B_{xs} \sinh(k_u x_0)\, B_{ys} \cosh(k_u y_0) \cos(\delta) + B_{yc} \sinh(k_u y_0)\, B_{xc} \cosh(k_u x_0) \cos(\delta) - B_{yc} \sinh(k_u y_0)\, B_{xs} \sinh(k_u x_0) \sin(\delta) + B_{ys} \cosh(k_u y_0)\, B_{xc} \cosh(k_u x_0) \sin(\delta) + B_{ys} \cosh(k_u y_0)\, B_{xs} \sinh(k_u x_0) \cos(\delta) \} \quad (116)$$

$$\left\langle \frac{dv_{y(2)}}{dz} \right\rangle = \frac{k_u c^2}{2 B_0^2 v_{z(0)}} \{ B_{xc} \sinh(k_u x_0)\, B_{yc} \cosh(k_u y_0) \cos(\delta) + B_{xc} \sinh(k_u x_0)\, B_{ys} \sinh(k_u y_0) \sin(\delta) - B_{xs} \cosh(k_u x_0)\, B_{yc} \cosh(k_u y_0) \sin(\delta) + B_{xs} \cosh(k_u x_0)\, B_{ys} \sinh(k_u y_0) \cos(\delta) - B_{yc} \cosh(k_u y_0)\, B_{xc} \sinh(k_u x_0) \cos(\delta) + B_{yc} \cosh(k_u y_0)\, B_{xs} \cosh(k_u x_0) \sin(\delta) - B_{yc} \cosh(k_u y_0)\, B_{yc} \sinh(k_u y_0) - B_{ys} \sinh(k_u y_0)\, B_{xc} \sinh(k_u x_0) \sin(\delta) - B_{ys} \sinh(k_u y_0)\, B_{xs} \cosh(k_u x_0) \cos(\delta) - B_{ys} \sinh(k_u y_0)\, B_{ys} \cosh(k_u y_0) \} \quad (117)$$

$$\left\langle \frac{dv_{z(2)}}{dz} \right\rangle = \frac{k_u c^2}{2B_0^2 v_{z(0)}} \{-B_{yc}\cosh(k_u y_0) B_{ys}\sinh(k_u y_0) + B_{ys}\sinh(k_u y_0) B_{yc}\cosh(k_u y_0) - B_{xc}\cosh(k_u x_0) B_{xs}\sinh(k_u x_0) + B_{xs}\sinh(k_u x_0) B_{xc}\cosh(k_u x_0)\} \quad (118)$$

Consider particles near $x_0 = 0$ and $y_0 = 0$, i.e. with $k_u x_0 \ll 1$ and $k_u y_0 \ll 1$. Expanding in $k_u x_0$ and $k_u x_0$ to first order, we have

$$\left\langle \frac{dv_{x(2)}}{dz} \right\rangle = \frac{k_u c^2}{2B_0^2 v_{z(0)}} \{-B_{xc}B_{xc}k_u x_0 - B_{xc}B_{yc}k_u y_0 \cos(\delta) - B_{xc}B_{ys}\sin(\delta) - B_{xs}k_u x_0 B_{xs} + B_{xs}k_u x_0 B_{yc}k_u y_0 \sin(\delta) - B_{xs}k_u x_0 B_{ys}\cos(\delta) + B_{yc}k_u y_0 B_{xc}\cos(\delta) - B_{yc}k_u y_0 B_{xs}k_u x_0 \sin(\delta) + B_{ys}B_{xc}\sin(\delta) + B_{ys}B_{xs}k_u x_0 \cos(\delta)\} \quad (119)$$

$$\left\langle \frac{dv_{y(2)}}{dz} \right\rangle = \frac{k_u c^2}{2B_0^2 v_{z(0)}} \{B_{xc}k_u x_0 B_{yc}\cos(\delta) + B_{xc}k_u x_0 B_{ys}k_u y_0 \sin(\delta) - B_{xs}B_{yc}\sin(\delta) + B_{xs}B_{ys}k_u y_0 \cos(\delta) - B_{yc}B_{xc}k_u x_0 \cos(\delta) + B_{yc}B_{xs}\sin(\delta) - B_{yc}B_{yc}k_u y_0 - B_{ys}k_u y_0 B_{xc}k_u x_0 \sin(\delta) - B_{ys}k_u y_0 B_{xs}\cos(\delta) - B_{ys}k_u y_0 B_{ys}\} \quad (120)$$

$$\left\langle \frac{dv_{z(2)}}{dz} \right\rangle = \frac{k_u c^2}{2B_0^2 v_{z(0)}} \{-B_{yc}B_{ys}k_u y_0 + B_{ys}k_u y_0 B_{yc} - B_{xc}B_{xs}k_u x_0 + B_{xs}k_u x_0 B_{xc}\} \quad (121)$$

To first order in $k_u x_0$ and $k_u x_0$, these equation simplify to

$$\left\langle \frac{dv_{x(2)}}{dz} \right\rangle = \frac{k_u c^2}{2B_0^2 v_{z(0)}} \{k_u x_0[-B_{xc}B_{xc} - B_{xs}B_{xs}]\} \quad (122)$$

$$\left\langle \frac{dv_{y(2)}}{dz} \right\rangle = \frac{k_u c^2}{2B_0^2 v_{z(0)}} \{k_u y_0[-B_{yc}B_{yc} - B_{ys}B_{ys}]\} \quad (123)$$

$$\left\langle \frac{dv_{z(2)}}{dz} \right\rangle = 0 \quad (124)$$

A final simplification yields

$$\left\langle \frac{dv_{x(2)}}{dz} \right\rangle = -\frac{k_u^2 c^2}{2v_{z(0)}} x_0 \quad (125)$$

$$\left\langle \frac{dv_{y(2)}}{dz} \right\rangle = -\frac{k_u^2 c^2}{2v_{z(0)}} y_0 \quad (126)$$

$$\left\langle \frac{dv_{z(2)}}{dz} \right\rangle = 0 \quad (127)$$

The second order velocities through the undulator are

$$
\begin{aligned}
v_{x(2)} &= -\frac{k_u^2 c^2}{2v_{z(0)}} x_0 z + osc(2k_u z) && (128)\\
v_{y(2)} &= -\frac{k_u^2 c^2}{2v_{z(0)}} y_0 z + osc(2k_u z) && (129)\\
v_{z(2)} &= osc(2k_u z) && (130)
\end{aligned}
$$

where $osc(2k_u z)$ represent terms that go as $\cos(2k_u z)$ or $\sin(2k_u z)$.

3.3 Solution To The Equations Of Motion

The velocity of a particle in the undulator can now be given to second order. Using

$$
\begin{aligned}
v_x &= v_{x(0)} + \epsilon v_{x(1)} + \epsilon^2 v_{x(2)} + \cdots && (131)\\
v_y &= v_{y(0)} + \epsilon v_{y(1)} + \epsilon^2 v_{y(2)} + \cdots && (132)\\
v_z &= v_{z(0)} + \epsilon v_{z(1)} + \epsilon^2 v_{z(2)} + \cdots && (133)
\end{aligned}
$$

with

$$
\epsilon = \frac{qB_0}{\gamma m k_u c} \qquad (134)
$$

we get

$$
\begin{aligned}
v_x &= -\frac{q}{\gamma m k_u}\left[B_{yc}\cosh\left(k_u y_0\right)\sin\left(k_u z + \delta\right) - B_{ys}\sinh\left(k_u y_0\right)\cos\left(k_u z + \delta\right)\right]\\
&\quad -\frac{q^2 B_0^2}{2\gamma^2 m^2 v_{z0}} x_0 z + \epsilon^2 osc(2k_u z) + \cdots && (135)\\
v_y &= \frac{q}{\gamma m k_u}\left[B_{xc}\cosh\left(k_u x_0\right)\sin\left(k_u z\right) - B_{xs}\sinh\left(k_u x_0\right)\cos\left(k_u z\right)\right]\\
&\quad -\frac{q^2 B_0^2}{2\gamma^2 m^2 v_{z0}} y_0 z + \epsilon^2 osc(2k_u z) + \cdots && (136)\\
v_z &= v_{z0} + \epsilon^2 osc(2k_u z) + \cdots && (137)
\end{aligned}
$$

The average velocity of the particle in the undulator is

$$
\begin{aligned}
\langle v_x \rangle &= -\frac{q^2 B_0^2}{2\gamma^2 m^2 v_{z0}} x_0 z && (138)\\
\langle v_y \rangle &= -\frac{q^2 B_0^2}{2\gamma^2 m^2 v_{z0}} y_0 z && (139)\\
\langle v_z \rangle &= v_{z0} && (140)
\end{aligned}
$$

3.4 Laboratory Coordinate System

The undulator frame is rotated by 45° in the laboratory frame. The transformation to the laboratory frame coordinates is

$$
\begin{aligned}
x_L &= \frac{1}{\sqrt{2}}\left(x - y\right) && (141)\\
y_L &= \frac{1}{\sqrt{2}}\left(x + y\right) && (142)
\end{aligned}
$$

The transformation of the average velocity is

$$\langle v_{xL} \rangle = \frac{1}{\sqrt{2}} \left(\langle v_x \rangle - \langle v_y \rangle \right) \tag{143}$$

$$\langle v_{yL} \rangle = \frac{1}{\sqrt{2}} \left(\langle v_x \rangle + \langle v_y \rangle \right) \tag{144}$$

Using equations 138 and 139, the average velocities in the laboratory frame are

$$\langle v_{xL} \rangle = -\frac{q^2 B_0^2}{2\gamma^2 m^2 v_{z0}} x_{L0} z \tag{145}$$

$$\langle v_{yL} \rangle = -\frac{q^2 B_0^2}{2\gamma^2 m^2 v_{z0}} y_{L0} z \tag{146}$$

4 Undulator Focusing

If the undulator has length L, the slope of the trajectory in the laboratory frame when the beam leaves the undulator is

$$\left\langle \frac{dx_L}{dz} \right\rangle = -\frac{q^2 B_0^2 L}{2\gamma^2 m^2 v_{z0}^2} x_{L0} \tag{147}$$

$$\left\langle \frac{dy_L}{dz} \right\rangle = -\frac{q^2 B_0^2 L}{2\gamma^2 m^2 v_{z0}^2} y_{L0} \tag{148}$$

If we treat the undulator as a thin lens and represent the exit slope as

$$\left\langle \frac{dx_L}{dz} \right\rangle = -\frac{x_{L0}}{f_x} \tag{149}$$

$$\left\langle \frac{dy_L}{dz} \right\rangle = -\frac{y_{L0}}{f_y} \tag{150}$$

we see that the focal length of the undulator is

$$f_x = f_y = \frac{2\gamma^2 m^2 v_{z0}^2}{q^2 B_0^2 L} \tag{151}$$

The focal length is the same in both the x and y directions. The focal length is independent of the mode of the undulator and is independent of the K value.

5 Conclusion

Under the assumption that the transverse rolloff of the quadrant fields are negligible in the vicinity of the beam, the Delta undulator focuses the beam in both the x and the y directions with focal length

$$f_x = f_y = \frac{2\gamma^2 m^2 v_{z0}^2}{q^2 B_0^2 L} \tag{152}$$

This is independent of the mode of the undulator and independent of the K value.

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