

Setting The K Value And Polarization Mode Of The Delta Undulator

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Abstract

This note provides the details for setting the longitudinal positions of the four quadrants of the Delta undulator in order to produce the desired wavelength and polarization mode of the emitted light.

1 Introduction¹

The longitudinal positions of the four quadrants of the Delta undulator determine the characteristics of the light it produces for given electron beam parameters. In this note, we study the connection between the quadrant positions and the light characteristics.

The note breaks into parts the study of how the quadrant positions determine the light characteristics. First, the relations between quadrant positions and the magnetic field in the undulator are determined. Then the relation between the magnetic field in the undulator and the electric field of the light is derived. After that, the relation between the electric field of the light and the polarization ellipse is derived. The relation between electric field and the Stokes parameters is also derived. Finally, the relation between the polarization ellipse and the Stokes parameters is derived.

2 Overview

2.1 Parameters

The Delta undulator has 4 movable quadrants. An overall shift of all 4 quadrants produces a phase shift of the electron beam relative to a light wave. This feature will not be used since a separate phase shifter will be placed before the undulator. So three parameters, three relative quadrant positions, must be determined in order to set the K value and polarization mode of the undulator. The fourth quadrant position is set by making the average of the quadrant positions equal to zero.

The three quadrant positions determine three parameters of the magnetic field in the undulator. The magnetic field is equivalent to the field from two crossed planar adjustable phase undulators, as will be shown below. The three parameters of the magnetic field are the K value relative to the maximum value K_0 , the ratio of the field strengths from the two crossed planar undulators, and the phase difference of the fields in the two crossed undulators. The K value determines the wavelength of light produced by the undulator. The relative strengths of the two crossed planar undulators and the phase difference of the magnetic fields determine the light polarization characteristics. The

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required value of K_0 is determined experimentally, and so it is a parameter put in by hand after measurement.

A charged particle going through the undulator produces light. The electric field from the light is defined by three parameters. The first is the wavelength of the light. Next is the ratio of the transverse electric field strengths in two perpendicular directions. The last parameter is the phase difference between the electric field components in the two perpendicular directions. The combination of the ratio of the field strengths and the phase difference determine the polarization characteristics of the light. The phase difference also determines the rotation direction of the electric field in a transverse plane at a fixed location as time evolves. It also determines the helicity of the electric field wave, how it rotates in space along the direction of propagation at fixed time.

The electric field polarization can be characterized by the polarization ellipse. The ellipse is characterized by the ratio of the minor and major axes, the angle of the ellipse relative to an axis of the transverse plane, and the rotation direction of the field around the ellipse. The rotation direction is required to resolve a sign ambiguity of the phase difference between the components of the electric field, which the angle of the ellipse and the ratio of the axes does not give. In essence, the polarization ellipse plus the rotation direction (field helicity) gives two parameters describing the field, namely the ratio of the field components and the phase between them. The third parameter required to describe the light is the wavelength, which must be measured separately.

The electric field can also be characterized by the Stokes parameters. There are four Stokes parameters and three of them are independent. We use ratios of the parameters to determine the polarization mode. Three ratios are calculated, but they are related by a constraint. The third ratio is used, even with the constraint, to resolve a sign ambiguity. The ratios of the Stokes parameters plus the wavelength of the light give three parameters needed to characterize the electric field.

2.2 Relative Phases

A topic which will come up repeatedly is the rotation direction of a vector as a parameter w evolves. Suppose the components of the vector are given by

$$x = A \cos(w) \tag{1}$$

$$y = B \cos(w + \varphi) \tag{2}$$

where φ is the phase of y relative to x . In the x-y plane, as w increases, the point with coordinates (x, y) rotates clockwise when $0 < \varphi < \pi$, and the point with coordinates (x, y) rotates counterclockwise when $-\pi < \varphi < 0$. This is easily seen by looking at the initial step at $w = 0$ for a positive increment δw .

$$\delta x = 0 \tag{3}$$

$$\delta y = -B \sin(\varphi) \delta w \tag{4}$$

δy is negative, giving clockwise rotation, when $\sin(\varphi) > 0$, or when $0 < \varphi < \pi$. δy is positive, giving counterclockwise rotation, when $\sin(\varphi) < 0$, or when $-\pi < \varphi < 0$.

We adopt the convention that phase angles and geometrical angles are in the range $[-\pi, \pi]$. Imposing this restriction will give unique values to all quantities, whether we are calculating radiation properties from row positions, or whether we are calculating row positions from radiation properties.

We now go through in detail the steps leading from setting the quadrant positions to determining the radiation properties.

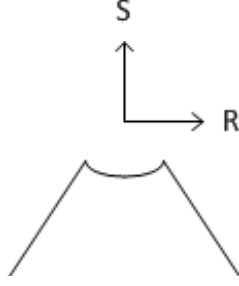


Figure 1: Coordinate system for the scalar potential of a single quadrant.

3 Magnetic Field On The Undulator Beam Axis

3.1 Scalar Potential For The Undulator

In a previous technical note² the scalar potential from a single magnet array was derived. In the coordinate system shown in figure 1, the scalar potential has the form

$$\phi = \phi_0 \exp(-k_u s) \cos(k_u(z - z_0)) \quad (5)$$

where ϕ_0 is a constant, $k_u = 2\pi/\lambda_u$ where λ_u is the undulator period, z is the coordinate down the undulator axis, and z_0 gives the quadrant position along z . We work close to the beam axis where the variation of the scalar potential in the r -direction is small, and we ignore the r dependence.

We use this form of the potential to calculate the magnetic scalar potential in the undulator by rotating the four quadrants and their scalar potentials. The Delta is positioned as shown on the left side of figure 2. In the laboratory, z is along the beam direction, y_L is up, and x_L makes a

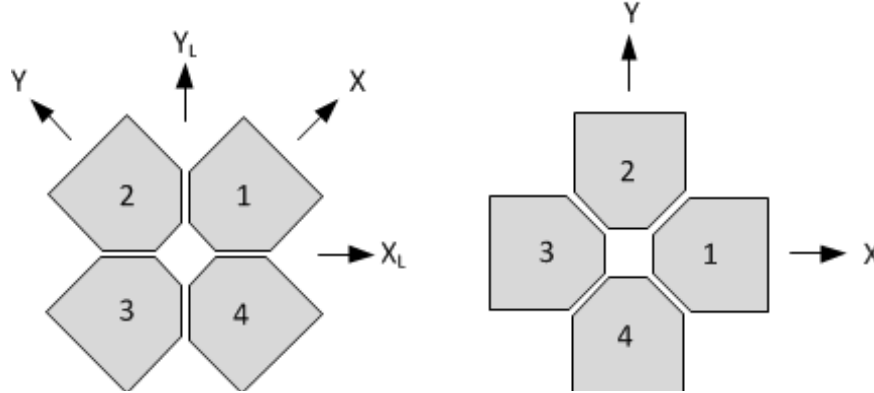


Figure 2: The left side of the figure shows the Delta undulator in its configuration in the tunnel where y_L is up, z is in the beam direction, and x_L makes a right handed system. For our calculations, we use the rotated coordinate system on the right, where x is along the line pointing from quadrant 3 to quadrant 1, y is along the line pointing from quadrant 4 to quadrant 2, and z makes a right handed system.

right handed system. For our calculations, it is more convenient to use the rotated system x, y, z ,

²Z. Wolf, "A Calculation Of The Fields In The Delta Undulator", LCLS-TN-14-1, January, 2014.

where x is in the direction from quadrant 3 to quadrant 1, y is in the direction from quadrant 4 to quadrant 2, and z is in the beam direction. Using equation 5, the scalar potential for each of the quadrants in the x, y, z system is

$$\phi_1(x, y, z) = \phi_{0Q} \exp(k_u x) \cos(k_u (z - z_{01})) \quad (6)$$

$$\phi_2(x, y, z) = \phi_{0Q} \exp(k_u y) \cos(k_u (z - z_{02})) \quad (7)$$

$$\phi_3(x, y, z) = -\phi_{0Q} \exp(-k_u x) \cos(k_u (z - z_{03})) \quad (8)$$

$$\phi_4(x, y, z) = -\phi_{0Q} \exp(-k_u y) \cos(k_u (z - z_{04})) \quad (9)$$

where z_{0i} is the longitudinal shift of quadrant i , and ϕ_{0Q} is the amplitude of the scalar potential of all the identical quadrants on the axis of the undulator where $x = 0$ and $y = 0$. Quadrants 3 and 4 are loaded with opposite polarity magnets as quadrants 1 and 2 in order to make a vertical field planar undulator in the laboratory frame when all the rows are aligned. This accounts for the minus signs, $-\phi_{0Q}$, in the potentials for quadrants 3 and 4. In order for the field from each quadrant to go through the relative phase range $[-\pi, \pi]$, the z_{0i} must have the range $[-\frac{\lambda_u}{2}, \frac{\lambda_u}{2}]$. Changing z_{0i} by a multiple of λ_u does not change the field from that quadrant. This can be used to restrict the z_{0i} to the range $[-\frac{\lambda_u}{2}, \frac{\lambda_u}{2}]$. We do this in order to determine unique quadrant positions from given radiation properties.

The scalar potential for the undulator is the sum of the scalar potentials for the quadrants. Quadrants 1 and 3 both depend on x , and quadrants 2 and 4 both depend on y . We first add the scalar potentials for quadrants 1 and 3, and then add the scalar potentials for quadrants 2 and 4, and then add the sums to get the scalar potential for the whole undulator. We will interpret this as forming the entire undulator from two crossed planar adjustable phase undulators.

The scalar potential for the combination of quadrants 1 and 3 is given by

$$\phi_{13} = \phi_{0Q} \exp(k_u x) \cos(k_u (z - z_{01})) - \phi_{0Q} \exp(-k_u x) \cos(k_u (z - z_{03})) \quad (10)$$

Let

$$z_{01} = Z_{13} + \frac{\Delta_{13}}{2} \quad (11)$$

$$z_{03} = Z_{13} - \frac{\Delta_{13}}{2} \quad (12)$$

So

$$Z_{13} = \frac{z_{01} + z_{03}}{2} \quad (13)$$

is the average z -position of the quadrants, and

$$\Delta_{13} = z_{01} - z_{03} \quad (14)$$

is the z -shift between the quadrants. With these definitions, the scalar potential for the pair of quadrants becomes

$$\begin{aligned} \phi_{13} = & 2\phi_{0Q} \sinh(k_u x) \cos\left(k_u \frac{\Delta_{13}}{2}\right) \cos(k_u (z - Z_{13})) \\ & + 2\phi_{0Q} \cosh(k_u x) \sin\left(k_u \frac{\Delta_{13}}{2}\right) \sin(k_u (z - Z_{13})) \end{aligned} \quad (15)$$

This is the scalar potential for a planar adjustable phase undulator³. The full range of amplitudes and phases are covered if the range of Δ_{13} includes $[-\frac{\lambda_u}{2}, \frac{\lambda_u}{2}]$ and the range of Z_{13} includes $[-\frac{\lambda_u}{2}, \frac{\lambda_u}{2}]$, both of which are covered by the range of the z_{0i} given above.

³Z. Wolf, "Variable Phase PPM Undulator Study", LCLS-TN-11-1, May, 2011.

Similarly, the scalar potential for the combination of quadrants 2 and 4 is given by

$$\phi_{24} = \phi_{0Q} \exp(k_u y) \cos(k_u(z - z_{02})) - \phi_{0Q} \exp(-k_u y) \cos(k_u(z - z_{04})) \quad (16)$$

Let

$$z_{02} = Z_{24} + \frac{\Delta_{24}}{2} \quad (17)$$

$$z_{04} = Z_{24} - \frac{\Delta_{24}}{2} \quad (18)$$

So

$$Z_{24} = \frac{z_{02} + z_{04}}{2} \quad (19)$$

is the average z-position of the quadrants, and

$$\Delta_{24} = z_{02} - z_{04} \quad (20)$$

is the z-shift between the quadrants.

With these definitions, the scalar potential for the pair of quadrants becomes

$$\begin{aligned} \phi_{24} &= 2\phi_{0Q} \sinh(k_u y) \cos\left(k_u \frac{\Delta_{24}}{2}\right) \cos(k_u(z - Z_{24})) \\ &\quad + 2\phi_{0Q} \cosh(k_u y) \sin\left(k_u \frac{\Delta_{24}}{2}\right) \sin(k_u(z - Z_{24})) \end{aligned} \quad (21)$$

This is again the potential for a planar adjustable phase undulator.

The scalar potential for the undulator is the sum of the scalar potentials for the quadrant pairs.

$$\phi = \phi_{13} + \phi_{24} \quad (22)$$

Performing the sum, we find

$$\begin{aligned} \phi &= 2\phi_{0Q} \sinh(k_u x) \cos\left(k_u \frac{\Delta_{13}}{2}\right) \cos(k_u(z - Z_{13})) \\ &\quad + 2\phi_{0Q} \cosh(k_u x) \sin\left(k_u \frac{\Delta_{13}}{2}\right) \sin(k_u(z - Z_{13})) \\ &\quad + 2\phi_{0Q} \sinh(k_u y) \cos\left(k_u \frac{\Delta_{24}}{2}\right) \cos(k_u(z - Z_{24})) \\ &\quad + 2\phi_{0Q} \cosh(k_u y) \sin\left(k_u \frac{\Delta_{24}}{2}\right) \sin(k_u(z - Z_{24})) \end{aligned} \quad (23)$$

By putting in the various values for Z_{13} , Z_{24} , Δ_{13} , and Δ_{24} , we get the scalar potential for the various undulator modes at different K values.

3.2 Magnetic Field On The Undulator Axis

The magnetic field in the undulator is given by $B = \nabla\phi$. Taking the gradient and setting $x = 0$ and $y = 0$ on the undulator axis, we find

$$B_x(0, 0, z) = 2\phi_{0Q} k_u \cos\left(k_u \frac{\Delta_{13}}{2}\right) \cos(k_u(z - Z_{13})) \quad (24)$$

$$B_y(0, 0, z) = 2\phi_{0Q} k_u \cos\left(k_u \frac{\Delta_{24}}{2}\right) \cos(k_u(z - Z_{24})) \quad (25)$$

If we let $B_0 = 2\phi_{0Q}k_u$, then on the undulator axis

$$B_x = B_0 \cos\left(k_u \frac{\Delta_{13}}{2}\right) \cos(k_u(z - Z_{13})) \quad (26)$$

$$B_y = B_0 \cos\left(k_u \frac{\Delta_{24}}{2}\right) \cos(k_u(z - Z_{24})) \quad (27)$$

In order to simplify these formulas further, let

$$B_{x0} = B_0 \cos\left(k_u \frac{\Delta_{13}}{2}\right) \quad (28)$$

$$B_{y0} = B_0 \cos\left(k_u \frac{\Delta_{24}}{2}\right) \quad (29)$$

$$\phi_{x0} = -k_u Z_{13} \quad (30)$$

$$\phi_{y0} = -k_u Z_{24} \quad (31)$$

Note that positive and negative values of Δ_{13} and Δ_{24} produce the same field amplitude because of the cosine dependence. With these substitutions, the fields become

$$B_x = B_{x0} \cos(k_u z + \phi_{x0}) \quad (32)$$

$$B_y = B_{y0} \cos(k_u z + \phi_{y0}) \quad (33)$$

We adopt the convention that field amplitudes are positive, which requires that we restrict the range of Δ_{13} and Δ_{24} to $[-\frac{\lambda_u}{2}, \frac{\lambda_u}{2}]$. Each z_{0i} has the range $[-\frac{\lambda_u}{2}, \frac{\lambda_u}{2}]$, however, giving a possible range to Δ_{ij} of $[-\lambda_u, \lambda_u]$. Suppose $\Delta_{ij} > \frac{\lambda_u}{2}$ so $\cos\left(k_u \frac{\Delta_{ij}}{2}\right)$ is negative. Subtract $\frac{\lambda_u}{2}$ from z_{0i} and add $\frac{\lambda_u}{2}$ to z_{0j} . This changes Δ_{ij} by $-\lambda_u$, which changes the sign of $\cos\left(k_u \frac{\Delta_{ij}}{2}\right)$, making it positive, and does not change the value of Z_{ij} . If $\Delta_{ij} < -\frac{\lambda_u}{2}$ so $\cos\left(k_u \frac{\Delta_{ij}}{2}\right)$ is negative, add $\frac{\lambda_u}{2}$ to z_{0i} and subtract $\frac{\lambda_u}{2}$ from z_{0j} . This changes Δ_{ij} by $+\lambda_u$, which changes the sign of $\cos\left(k_u \frac{\Delta_{ij}}{2}\right)$, making it positive.

We can change the origin of z so that B_x has zero phase. In this case, we write the fields as

$$B_x = B_{x0} \cos(k_u z) \quad (34)$$

$$B_y = B_{y0} \cos(k_u z + \delta) \quad (35)$$

$$\delta = \phi_{y0} - \phi_{x0} = k_u (Z_{13} - Z_{24}) \quad (36)$$

As noted previously, we adopt the convention that δ be in the range $[-\pi, \pi]$. Multiples of 2π can be added to or subtracted from δ . This amounts to changing both the Z_{ij} , one by $\frac{\lambda_u}{2}$ and the other by $-\frac{\lambda_u}{2}$, so the difference changes by λ_u . We do this by changing z_{0i} and z_{0j} both by $\frac{\lambda_u}{2}$ so Δ_{ij} doesn't change. If $\delta > \pi$, or $Z_{13} - Z_{24} > \frac{\lambda_u}{2}$, subtract $\frac{\lambda_u}{2}$ from z_{01} and z_{03} and add $\frac{\lambda_u}{2}$ to z_{02} and z_{04} . This changes $Z_{13} - Z_{24}$ by $-\lambda_u$, which changes δ by -2π . If $\delta < -\pi$, or $Z_{13} - Z_{24} < -\frac{\lambda_u}{2}$, add $\frac{\lambda_u}{2}$ to z_{01} and z_{03} and subtract $\frac{\lambda_u}{2}$ from z_{02} and z_{04} . This changes $Z_{13} - Z_{24}$ by $+\lambda_u$, which changes δ by $+2\pi$. This places δ in the range $[-\pi, \pi]$. This convention for the range of δ will be used when we discuss the helicity of the magnetic field.

3.3 Magnetic Field Helicity

Using equations 34 and 35, we can plot the point (B_x, B_y) as one moves down the undulator. The point (B_x, B_y) traces out an ellipse as one moves in z . This is shown in figure 3 for equal strength

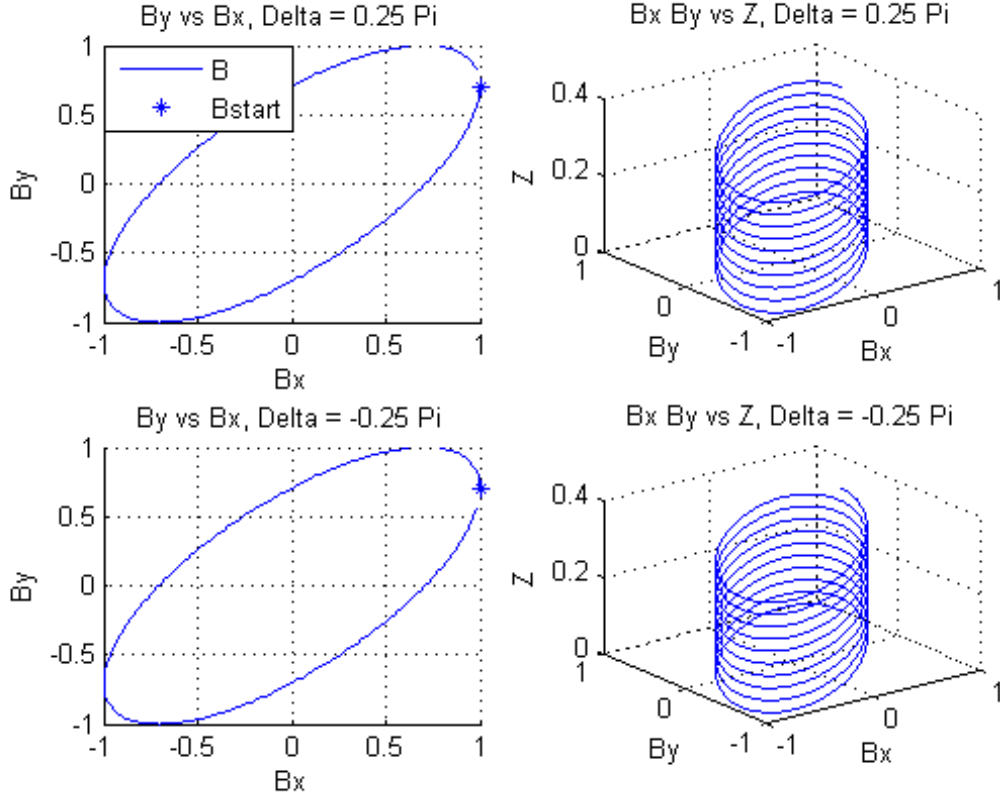


Figure 3: Plots of the point (B_x, B_y) as one moves down the undulator. Positive values of δ give negative helicity fields. Negative values of δ give positive helicity fields.

B_x and B_y , and $\delta = \pi/4$ and $\delta = -\pi/4$. In general, positive values of δ , $0 < \delta < \pi$, give clockwise rotation as z increases, giving left hand or negative helicity magnetic fields. Negative values of δ , $-\pi < \delta < 0$, give counterclockwise rotation as z increases, giving right hand or positive helicity magnetic fields.

3.4 Relation Between The Magnetic Field Parameters And The Quadrant Positions

Knowing the undulator quadrant positions lets us determine the magnetic field parameters. The K value of the undulator relative to the maximum value K_0 is given by

$$\frac{K}{K_0} = \frac{\sqrt{B_{x0}^2 + B_{y0}^2}}{\sqrt{B_0^2 + B_0^2}} \quad (37)$$

$$= \frac{1}{\sqrt{2}} \sqrt{\cos^2\left(k_u \frac{\Delta_{13}}{2}\right) + \cos^2\left(k_u \frac{\Delta_{24}}{2}\right)} \quad (38)$$

$$= \frac{1}{\sqrt{2}} \sqrt{\cos^2\left(k_u \frac{z_{01} - z_{03}}{2}\right) + \cos^2\left(k_u \frac{z_{02} - z_{04}}{2}\right)} \quad (39)$$

The ratio of the field strengths B_{y0}/B_{x0} is given by

$$\frac{B_{y0}}{B_{x0}} = \frac{\cos\left(k_u \frac{\Delta_{24}}{2}\right)}{\cos\left(k_u \frac{\Delta_{13}}{2}\right)} \quad (40)$$

$$= \frac{\cos\left(k_u \frac{z_{02} - z_{04}}{2}\right)}{\cos\left(k_u \frac{z_{01} - z_{03}}{2}\right)} \quad (41)$$

The phase of B_y relative to B_x is

$$\delta = k_u (Z_{13} - Z_{24}) \quad (42)$$

$$= \frac{k_u}{2} [(z_{01} + z_{03}) - (z_{02} + z_{04})] \quad (43)$$

Similarly, knowing the magnetic field parameters lets us determine the quadrant positions. We need, however, one additional constraint in order to determine the four quadrant positions. We choose the extra constraint to be that the average z -position of the quadrants is zero.

$$z_{01} + z_{02} + z_{03} + z_{04} = 0 \quad (44)$$

The three other equations are

$$z_{01} - z_{03} = \Delta_{13} \quad (45)$$

$$z_{02} - z_{04} = \Delta_{24} \quad (46)$$

$$z_{01} - z_{02} + z_{03} - z_{04} = \frac{2}{k_u} \delta \quad (47)$$

The row position differences Δ_{13} and Δ_{24} are found in terms of K/K_0 and B_{y0}/B_{x0} to be

$$\Delta_{13} = \frac{2}{k_u} \cos^{-1} \left(\sqrt{\frac{2 \left(\frac{K}{K_0}\right)^2}{1 + \left(\frac{B_{y0}}{B_{x0}}\right)^2}} \right) \quad (48)$$

$$\Delta_{24} = \frac{2}{k_u} \cos^{-1} \left(\sqrt{\frac{2 \left(\frac{K}{K_0}\right)^2 \left(\frac{B_{y0}}{B_{x0}}\right)^2}{1 + \left(\frac{B_{y0}}{B_{x0}}\right)^2}} \right) \quad (49)$$

There is a sign ambiguity in Δ_{13} and Δ_{24} since both positive and negative angles have the same cosine. Either sign can be chosen for Δ_{13} and Δ_{24} since changing the sign of Δ_{13} , for instance, only interchanges the values of z_{01} and z_{03} , and all the equations for the z_{0i} remain valid. The same is true for Δ_{24} . In order to make \cos^{-1} single valued, we take the range \cos^{-1} to be $[0, \pi]$, then Δ_{13} and Δ_{24} are positive.

There was an additional sign ambiguity when we took the square root in the argument of the \cos^{-1} function. We chose the positive sign. This restricts the range of \cos^{-1} to $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The range of Δ_{ij} is then $[-\frac{\lambda_u}{2}, \frac{\lambda_u}{2}]$. We have already seen that this is required in order for field amplitudes to be defined as positive.

The solution for the row positions in terms of the field parameters is

$$z_{01} = \frac{1}{2} \left(\frac{\delta}{k_u} + \Delta_{13} \right) \quad (50)$$

$$z_{02} = \frac{1}{2} \left(-\frac{\delta}{k_u} + \Delta_{24} \right) \quad (51)$$

$$z_{03} = \frac{1}{2} \left(\frac{\delta}{k_u} - \Delta_{13} \right) \quad (52)$$

$$z_{04} = \frac{1}{2} \left(-\frac{\delta}{k_u} - \Delta_{24} \right) \quad (53)$$

where Δ_{13} and Δ_{24} are expressed in terms of the field parameters in equations 48 and 49. With Δ_{ij} in the range $[-\frac{\lambda_u}{2}, \frac{\lambda_u}{2}]$ and δ in the range $[-\pi, \pi]$, the range of the z_{0i} is $[-\frac{\lambda_u}{2}, \frac{\lambda_u}{2}]$.

4 Radiation Field

4.1 Electric Field Of The Radiation

In the previous section we found the magnetic field on the undulator axis in terms of the row positions of the four quadrants. We now find the electric field from a charge moving down the axis of the undulator and being accelerated by the magnetic field. The general form for the far electric field from an accelerating charge is⁴

$$\mathbf{E}(t) = \frac{q}{4\pi\epsilon_0 c} \left[\frac{\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{r(1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} \right]_{ret} \quad (54)$$

where q is the charge of the radiating particle, \mathbf{n} is the direction from the particle to the observation point, $\boldsymbol{\beta}$ is the velocity of the particle divided by the speed of light, and r is the distance from the particle to the observation point. The quantity in the bracket is evaluated at the retarded time t_r , defined such that $t = t_r + r(t_r)/c$.

We want the electric field at the fundamental frequency from an electron in a long undulator. To find the field, we take the Fourier transform of equation 54. The Fourier transform is⁵

$$\mathbf{E}(\omega) = \frac{i\omega q}{\sqrt{2\pi}4\pi\epsilon_0 c r} \int_{-\infty}^{\infty} [\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta}(t_r))] e^{-i\omega(t_r + \frac{r(t_r)}{c})} dt_r \quad (55)$$

In this expression for the Fourier transform, we assume a long undulator which lets us integrate from $t_r = -\infty$ to $+\infty$. This simplifies the calculations without affecting the resulting polarization mode of the field. We also assume that in the undulator, the particle stays very close to the beam axis, so the observation point can be taken in the forward z-direction so that

$$\mathbf{n} = \mathbf{e}_z \quad (56)$$

where \mathbf{e}_z is a constant unit vector in the z-direction. Furthermore, we assume that the observation point is far away so r can be taken as constant in the denominator. This assumption is not made in the phase term in the exponent, however.

With $\mathbf{n} = \mathbf{e}_z$, the triple cross product in the integral is given by

$$\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta}) = -\mathbf{e}_x \beta_x - \mathbf{e}_y \beta_y = -\boldsymbol{\beta} \quad (57)$$

⁴A. Hofmann, *The Physics of Synchrotron Radiation*, Cambridge University Press, Cambridge, 2004.

⁵*Ibid.*

To find $\boldsymbol{\beta}$, we start with the Lorentz force law

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{v} \times \mathbf{B}) \quad (58)$$

where $\mathbf{p} = \gamma m \mathbf{v}$, $\gamma = (1 - \beta^2)^{-1}$, and m is the particle rest mass. The energy of the particle is constant in the magnetic field of the undulator, except for radiation losses which we neglect. With constant energy, γ is constant. The Lorentz force law gives

$$\dot{\boldsymbol{\beta}} = \frac{q}{\gamma m} (\boldsymbol{\beta} \times \mathbf{B}) \quad (59)$$

Using

$$B_x = B_{x0} \cos(k_u z + \phi_{x0}) \quad (60)$$

$$B_y = B_{y0} \cos(k_u z + \phi_{y0}) \quad (61)$$

we find

$$\beta_x = -\frac{qB_{y0}}{\gamma m c k_u} \sin(k_u z + \phi_{y0}) \quad (62)$$

$$\beta_y = \frac{qB_{x0}}{\gamma m c k_u} \sin(k_u z + \phi_{x0}) \quad (63)$$

To perform the integral in the Fourier transform, we need to know z as a function of the retarded time. In order to simplify the analysis, we take the z -position of the charge at time t to be $z = v_z t$, where v_z is the average velocity in the z -direction, and we ignore small forward velocity deviations. This means that the z -position of the charge at time t_r is $z = v_z t_r$.

In the exponent of the Fourier transform, we need the distance r from the observation point to the charge. Let z_o be the z -position of the observation point. Then $r(t_r) = z_o - z(t_r) = z_o - v_z t_r$.

Inserting these expressions into the Fourier transform formula and performing the integral gives

$$\begin{aligned} \mathbf{E}(\omega) = & \frac{-q^2}{\sqrt{2\pi} 4\epsilon_0 \gamma m c^2 k_u r} \frac{1}{\omega} e^{-i\omega z_o/c} \\ & \times \{ \delta(k_u v_z - \omega(1 - \frac{v_z}{c})) [-\mathbf{e}_x B_{y0} e^{i\phi_{y0}} + \mathbf{e}_y B_{x0} e^{i\phi_{x0}}] \\ & - \delta(k_u v_z + \omega(1 - \frac{v_z}{c})) [-\mathbf{e}_x B_{y0} e^{-i\phi_{y0}} + \mathbf{e}_y B_{x0} e^{-i\phi_{x0}}] \} \end{aligned} \quad (64)$$

To find the electric field as a function of time, we perform the inverse Fourier transform given by

$$\mathbf{E}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{E}(\omega) e^{i\omega t} d\omega \quad (65)$$

To do the integral, we make use of the identity

$$f(x)\delta(ax - b) = \frac{1}{|a|} f\left(\frac{b}{a}\right) \quad (66)$$

We find

$$\begin{aligned} \mathbf{E}(t) = & \frac{q^2}{4\pi\epsilon_0 \gamma m c^2 r} \frac{1}{(1 - \frac{v_z}{c})^2} \\ & \times \{ \mathbf{e}_x B_{y0} \cos\left(\left(t - \frac{z_o}{c}\right) \left(\frac{k_u v_z}{1 - \frac{v_z}{c}}\right) + \phi_{y0}\right) \\ & - \mathbf{e}_y B_{x0} \cos\left(\left(t - \frac{z_o}{c}\right) \left(\frac{k_u v_z}{1 - \frac{v_z}{c}}\right) + \phi_{x0}\right) \} \end{aligned} \quad (67)$$

From this formula, we see that the radiation angular frequency ω_r and wave number k_r are

$$\omega_r = k_r c = \frac{k_u v_z}{\left(1 - \frac{v_z}{c}\right)} \quad (68)$$

where where $k_r = 2\pi/\lambda_r$ and λ_r is the radiation wavelength. Substituting the values for the angular frequency and wave number gives

$$\begin{aligned} \mathbf{E}(t) &= \frac{q^2}{4\pi\epsilon_0\gamma mc^2} \frac{1}{r} \frac{v_z}{\left(1 - \frac{v_z}{c}\right)^2} \\ &\times \{ \mathbf{e}_x B_{y0} \cos(\omega_r t - k_r z_o + \phi_{y0}) - \mathbf{e}_y B_{x0} \cos(\omega_r t - k_r z_o + \phi_{x0}) \} \end{aligned} \quad (69)$$

From equation 68, we have

$$\frac{\frac{v_z}{c}}{\left(1 - \frac{v_z}{c}\right)} = \frac{k_r}{k_u} = \frac{\lambda_u}{\lambda_r} \quad (70)$$

Using this expression in the formula for $\mathbf{E}(t)$ gives

$$\mathbf{E}(t) = \frac{q^2}{4\pi\epsilon_0\gamma m v_z} \frac{1}{r} \left(\frac{\lambda_u}{\lambda_r}\right)^2 \{ \mathbf{e}_x B_{y0} \cos(\omega_r t - k_r z_o + \phi_{y0}) - \mathbf{e}_y B_{x0} \cos(\omega_r t - k_r z_o + \phi_{x0}) \} \quad (71)$$

Equation 68 is also derived by considering the light slipping ahead of the charge by one radiation wavelength per undulator period

$$c \frac{\lambda_u}{v_z} = \lambda_u + \lambda_r \quad (72)$$

which leads to equation 70.

Solving for $\left(1 - \frac{v_z}{c}\right)$ in terms of γ , β_x , and β_y for a relativistic particle gives

$$1 - \frac{v_z}{c} = \frac{1}{2} \left(\frac{1}{\gamma^2} + \beta_x^2 + \beta_y^2 \right) \quad (73)$$

Equations 62 and 63 for β_x and β_y can be expressed as

$$\beta_x = -\frac{K_x}{\gamma} \sin(k_u z + \phi_{y0}) \quad (74)$$

$$\beta_y = \frac{K_y}{\gamma} \sin(k_u z + \phi_{x0}) \quad (75)$$

where $K_x = \frac{qB_{y0}}{mck_u}$ and $K_y = \frac{qB_{x0}}{mck_u}$. K_x is the K parameter for the 2-4 planar undulator, and K_y is the K parameter for the 1-3 planar undulator. Averaging the velocities over time gives an expression for the average forward velocity of a charge in an undulator

$$1 - \frac{v_z}{c} = \frac{1}{2\gamma^2} \left[1 + \frac{1}{2} (K_x^2 + K_y^2) \right] \quad (76)$$

Setting $v_z \simeq c$ in the numerator of 70 and using 76 gives

$$\lambda_r = \frac{\lambda_u}{2\gamma^2} \left[1 + \frac{1}{2} (K_x^2 + K_y^2) \right] \quad (77)$$

This expression for the spontaneous radiation wavelength in an undulator is the same as the expression for the resonance condition in an FEL. This means that at resonance, the FEL radiation wavelength is the same as the spontaneous radiation wavelength.

With the expressions given above, the electric field of the light from a relativistic charge in an undulator is

$$\mathbf{E}(t) = \frac{q^2}{\pi\epsilon_0 mc r} \frac{1}{[1 + \frac{1}{2}(K_x^2 + K_y^2)]^2} \gamma^3 \times \{\mathbf{e}_x B_{y0} \cos(\omega_r t - k_r z_o + \phi_{y0}) - \mathbf{e}_y B_{x0} \cos(\omega_r t - k_r z_o + \phi_{x0})\} \quad (78)$$

We see that the electric field has the form

$$E_x = FB_{y0} \cos(\omega_r t - k_r z_o + \phi_{y0}) \quad (79)$$

$$E_y = -FB_{x0} \cos(\omega_r t - k_r z_o + \phi_{x0}) \quad (80)$$

where F is a common factor that depends on the K value of the undulator. Let $E_{x0} = FB_{y0}$ and $E_{y0} = FB_{x0}$. We drop the 'o' for the observation point and write z instead of z_o . We can also add π to the phase of E_y to account for the minus sign in front. E_{x0} and E_{y0} are positive magnitudes, just as B_{x0} and B_{y0} are positive magnitudes. We choose our time origin to make the phase of E_x equal to zero. With these substitutions, the electric field is written as

$$E_x = E_{x0} \cos(\omega_r t - k_r z) \quad (81)$$

$$E_y = E_{y0} \cos(\omega_r t - k_r z + \epsilon) \quad (82)$$

where $\epsilon = \phi_{x0} - \phi_{y0} + \pi$. Expressing the phase difference in terms of the phase difference of the magnetic fields, we get $\epsilon = -\delta + \pi$, and in terms of undulator row shifts, $\epsilon = -k_u(Z_{13} - Z_{24}) + \pi$. Note that multiples of 2π can be added or subtracted from ϵ . We use this to adopt the convention that ϵ is in the range $[-\pi, \pi]$. This convention will be useful when we discuss the helicity of the electric field.

At fixed z , the electric field evolves in time as

$$E_x = E_{x0} \cos(\omega_r t) \quad (83)$$

$$E_y = E_{y0} \cos(\omega_r t + \epsilon) \quad (84)$$

Positive ϵ produces an electric field which rotates clockwise. Negative ϵ produces an electric field which rotates counterclockwise.

At fixed time, the electric field behaves as

$$E_x = E_{x0} \cos(-k_r z) \quad (85)$$

$$E_y = E_{y0} \cos(-k_r z + \epsilon) \quad (86)$$

Taking the negative of the arguments of the cosines in these formulas gives

$$E_x = E_{x0} \cos(k_r z) \quad (87)$$

$$E_y = E_{y0} \cos(k_r z - \epsilon) \quad (88)$$

Positive ϵ produces a right handed wave. Negative ϵ produces a left handed wave. This will be shown in more detail below.

4.2 Electric Field Helicity

The helicity of the electric field is found by looking at how the field varies with z at fixed time. Equations 87 and 88 are used in figure 4 to show how the fields evolve in z . Note that positive values of ϵ give a right hand, or positive helicity field. Negative values of ϵ give a left hand, or negative helicity field.

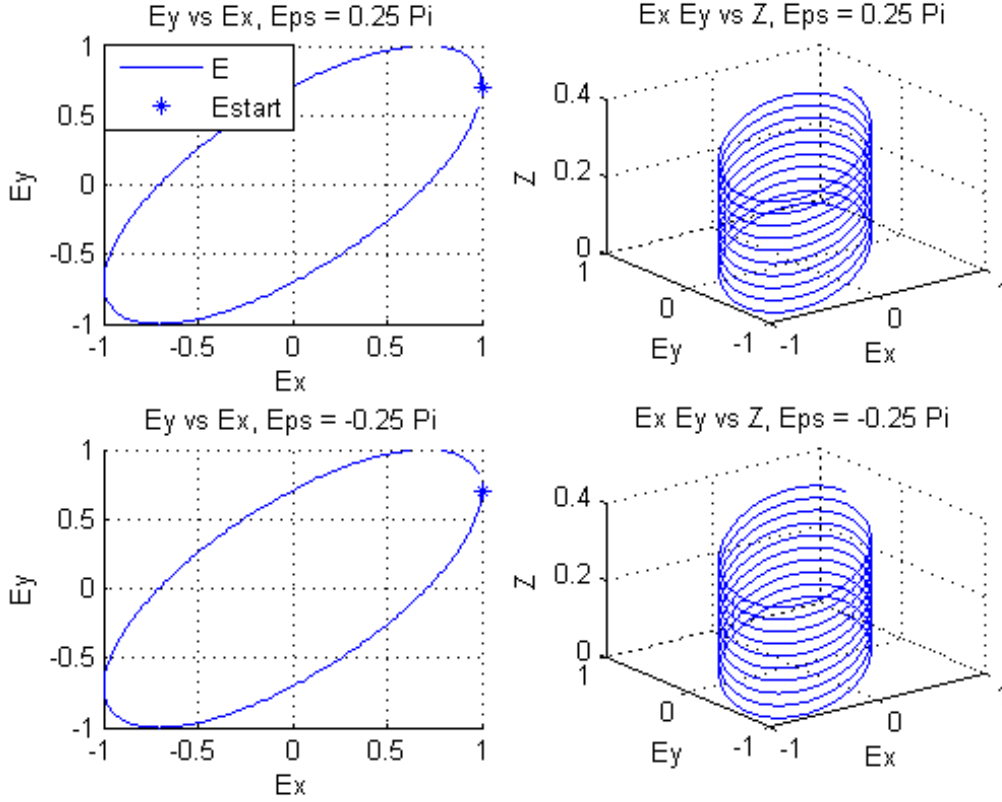


Figure 4: Positive values of ϵ give positive helicity fields. Negative values of ϵ give negative helicity fields.

The helicity of the electric field is opposite to the helicity of the undulator magnetic field. Since $\epsilon = -\delta + \pi$, we have

$$E_y = E_{y0} \cos(k_r z + \delta - \pi) \quad (89)$$

The extra π in the phase changes the helicity of the electric field compared to the magnetic field.

In order to gain insight why the helicity of the electric field is opposite to the helicity of the magnetic field, we start by considering the helicity of the motion of a charge in the magnetic field. The left part of figure 5 shows a charge in an undulator whose magnetic field has right handed helicity. The charge moves in z with right handed helicity, as we will show. The magnetic field given in equations 34 and 35 gives the charge velocity given in equations 62 and 63. The velocity can be rewritten as

$$\beta_x = \frac{qB_{y0}}{\gamma m c k_u} \cos(k_u z) \quad (90)$$

$$\beta_y = \frac{qB_{x0}}{\gamma m c k_u} \cos(k_u z - \delta - \pi) \quad (91)$$

The phase of β_y is the negative of the phase of the magnetic field B_y , but with an added π . So the helicity of the charge velocity is the same as the helicity of the magnetic field. This is expected since the force and the deflection direction are at right angles to the field and follow the field. A projection

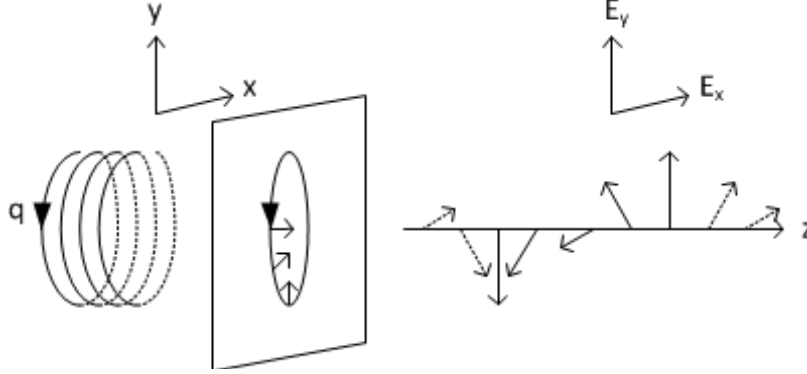


Figure 5: A charge in a magnetic field with right handed helicity produces an electric field with left handed helicity.

onto a screen of the charge motion shows the charge moving in a counterclockwise direction and tracing out an ellipse when viewed from $+z$ looking toward $-z$. This is shown in the middle of the figure. The charge acceleration at several locations is also shown.

The electric field from the charge is in the direction of the charge acceleration. This comes from the triple cross product in equation 54.

$$\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}] = -\dot{\boldsymbol{\beta}}(1 - \frac{v_z}{c}) \quad (92)$$

The right part of figure 5 shows the electric field from an electron as a function of z . When the electric field points away from us, it is marked with a dotted line, and when it points toward us, it is marked with a solid line. We start where the charge is moving down, and every eighth of a revolution we indicate the electric field direction in the charge acceleration direction. The key point is that because the velocity of light is greater than the charge velocity, the initial point we consider has its electric field at the greatest z location. Subsequent points, which are at larger z for the charge, are at smaller z for the electric field. This effectively reverses the z direction of the sequence of charge positions compared to electric field positions. This reverses the helicity of the electric field compared to the charge motion. The electric field is left handed as shown in the figure. In summary, the helicity of the electric field is opposite to the helicity of the undulator magnetic field.

4.3 Relation Between The Electric Field Parameters And The Magnetic Field Parameters

The parameters describing the electric field are the radiation wavelength λ_r , the ratio of the strength of E_y to E_x , namely E_{y0}/E_{x0} , and the phase ϵ of E_y relative to E_x . The radiation wavelength is given by equation 77. The term $K_x^2 + K_y^2$ is equal to K^2 given in equation 37. Using equation 37, we write the wavelength as

$$\lambda_r = \frac{\lambda_u}{2\gamma^2} \left[1 + \frac{1}{2} \left(\frac{K}{K_0} \right)^2 K_0^2 \right] \quad (93)$$

We measure K_0 during the initial magnetic measurements of the undulator. Once it is known, the wavelength is determined by the ratio K/K_0 .

Since we found that $E_{x0} = FB_{y0}$ and $E_{y0} = FB_{x0}$, we have

$$\frac{E_{y0}}{E_{x0}} = \frac{B_{x0}}{B_{y0}} \quad (94)$$

Finally, we have already found the relation between the phases

$$\epsilon = -\delta + \pi \quad (95)$$

Multiples of 2π can be added or subtracted from ϵ . We use this to adopt the convention that ϵ is in the range $[-\pi, \pi]$.

An additional parameter which is interesting is twice the time average value of the square of the electric field strength

$$I = 2 \langle \mathbf{E} \cdot \mathbf{E} \rangle \quad (96)$$

$$= E_{x0}^2 + E_{y0}^2 \quad (97)$$

The parameter I is proportional to a function of K and so is not an independent parameter.

$$I \propto \frac{K^2}{[1 + \frac{1}{2}K^2]^4} \quad (98)$$

I is proportional to the power of the radiation from the undulator.

The reverse transformation from electric field parameters to magnetic field parameters is found by solving the above equations for the magnetic field parameters.

$$\frac{K}{K_0} = \sqrt{\frac{2}{K_0^2} \left(\lambda_r \frac{2\gamma^2}{\lambda_u} - 1 \right)} \quad (99)$$

$$\frac{B_{y0}}{B_{x0}} = \frac{E_{x0}}{E_{y0}} \quad (100)$$

$$\delta = -\epsilon + \pi \quad (101)$$

Multiples of 2π can be added or subtracted from δ . We use this to adopt the convention that δ is in the range $[-\pi, \pi]$.

5 Polarization Ellipse

5.1 Equations Of The Polarization Ellipse

The polarization ellipse is found by considering the radiation wave at a fixed z location as a function of time. The equations for the electric field are

$$E_x = E_{x0} \cos(\omega_r t) \quad (102)$$

$$E_y = E_{y0} \cos(\omega_r t + \epsilon) \quad (103)$$

These equations can be expressed as a relation between E_x and E_y that does not involve time. This is the polarization ellipse⁶. Using the identity

$$\cos(\omega_r t + \epsilon) = \cos(\omega_r t) \cos(\epsilon) - \sin(\omega_r t) \sin(\epsilon) \quad (104)$$

and setting

$$\cos(\omega_r t) = \frac{E_x}{E_{x0}} \quad (105)$$

we find

$$\left(\frac{E_x}{E_{x0}} \right)^2 + \left(\frac{E_y}{E_{y0}} \right)^2 - 2 \left(\frac{E_x}{E_{x0}} \right) \left(\frac{E_y}{E_{y0}} \right) \cos(\epsilon) = \sin^2(\epsilon) \quad (106)$$

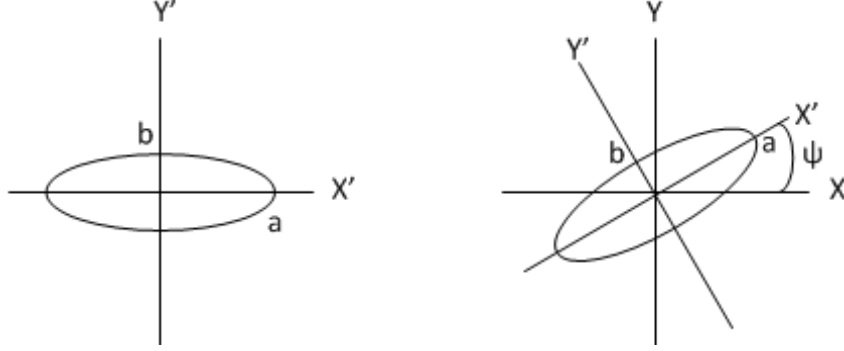


Figure 6: Ellipse with major axis a , minor axis b , rotated through an angle ψ .

This equation is analogous to the equation for a rotated ellipse as shown in figure 6. In the rotated primed coordinate system, the ellipse has equation

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1 \quad (107)$$

and in the unprimed system, the equation is

$$\begin{aligned} 1 &= x^2 \left(\frac{1}{a^2} \cos^2(\psi) + \frac{1}{b^2} \sin^2(\psi) \right) \\ &+ y^2 \left(\frac{1}{a^2} \sin^2(\psi) + \frac{1}{b^2} \cos^2(\psi) \right) \\ &+ 2xy \left(\frac{1}{a^2} - \frac{1}{b^2} \right) \cos(\psi) \sin(\psi) \end{aligned} \quad (108)$$

Because of the symmetry of the ellipse, we take ψ to be in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$. We can identify the major axis, minor axis, and rotation angle of the polarization ellipse through the following equations.

$$\frac{1}{E_{x0}^2 \sin^2(\epsilon)} = \left(\frac{1}{a^2} \cos^2(\psi) + \frac{1}{b^2} \sin^2(\psi) \right) \quad (109)$$

$$\frac{1}{E_{y0}^2 \sin^2(\epsilon)} = \left(\frac{1}{a^2} \sin^2(\psi) + \frac{1}{b^2} \cos^2(\psi) \right) \quad (110)$$

$$-\frac{\cos(\epsilon)}{E_{x0} E_{y0} \sin^2(\epsilon)} = \left(\frac{1}{a^2} - \frac{1}{b^2} \right) \cos(\psi) \sin(\psi) \quad (111)$$

The electric field amplitudes and phase required to produce a given polarization ellipse are given by solving for E_{x0} , E_{y0} , and ϵ in terms of a , b , and ψ . After some algebra, we find

$$E_{x0} = \sqrt{\frac{1}{2} [(a^2 + b^2) + (a^2 - b^2) \cos(2\psi)]} \quad (112)$$

$$E_{y0} = \sqrt{\frac{1}{2} [(a^2 + b^2) - (a^2 - b^2) \cos(2\psi)]} \quad (113)$$

$$\cos(\epsilon) = \frac{\sin(2\psi)}{|\sin(2\psi)|} \sqrt{\frac{(a^2 - b^2)^2}{a^4 + b^4 + a^2 b^2 (\tan^2 \psi + \cot^2 \psi)}} \quad (114)$$

⁶E. Hecht, *Optics*, 3rd edition, Addison Wesley Longman, Reading, Massachusetts, 1998.

Note that the ellipse parameters do not distinguish between positive and negative values of ϵ . The helicity of the field must be specified in addition to the ellipse parameters. For a right handed field, ϵ is positive. For a left handed field, ϵ is negative. A different ambiguity in the sign of $\cos(\epsilon)$ was resolved by using equation 111. The sign of $\cos(\epsilon)$ is the same as the sign of $\sin(2\psi)$ since $a > b$. So the \pm sign when taking the square root was replaced by $\sin(2\psi)/|\sin(2\psi)|$.

The polarization ellipse parameters can be found in terms of the electric field parameters. After some algebra, we find

$$\tan(2\psi) = 2 \frac{E_{x0}E_{y0}}{E_{x0}^2 - E_{y0}^2} \cos(\epsilon) \quad (115)$$

$$a^2 = \frac{1}{2} \left[E_{x0}^2 + E_{y0}^2 + \sqrt{(E_{x0}^2 - E_{y0}^2)^2 + 4E_{x0}^2 E_{y0}^2 \cos^2(\epsilon)} \right] \quad (116)$$

$$b^2 = \frac{1}{2} \left[E_{x0}^2 + E_{y0}^2 - \sqrt{(E_{x0}^2 - E_{y0}^2)^2 + 4E_{x0}^2 E_{y0}^2 \cos^2(\epsilon)} \right] \quad (117)$$

As noted above, the ellipse angle ψ does not depend on the sign of ϵ . The function $\tan(2\psi)$ has the same value for multiple values of ψ . We will resolve this ambiguity below when we solve for ψ . A sign ambiguity in front of the square root in a^2 and b^2 was resolved by making $a > b$.

5.2 Relation Between The Polarization Ellipse Parameters And The Electric Field Parameters

The polarization ellipse parameters useful for defining the undulator mode are the ratio b/a of minor axis to major axis, the direction ψ of the major axis relative to the x-axis, and the rotation direction of the field point around the ellipse.

The ratio b/a is given by

$$\frac{b^2}{a^2} = \frac{E_{x0}^2 + E_{y0}^2 - \sqrt{(E_{x0}^2 - E_{y0}^2)^2 + 4E_{x0}^2 E_{y0}^2 \cos^2(\epsilon)}}{E_{x0}^2 + E_{y0}^2 + \sqrt{(E_{x0}^2 - E_{y0}^2)^2 + 4E_{x0}^2 E_{y0}^2 \cos^2(\epsilon)}} \quad (118)$$

Expressed in terms of the field ratio parameter E_{y0}/E_{x0} and the relative phase ϵ , this relation becomes

$$\frac{b}{a} = \frac{\sqrt{1 + \frac{E_{y0}^2}{E_{x0}^2} - \sqrt{\left(1 - \frac{E_{y0}^2}{E_{x0}^2}\right)^2 + 4\frac{E_{y0}^2}{E_{x0}^2} \cos^2(\epsilon)}}}{\sqrt{1 + \frac{E_{y0}^2}{E_{x0}^2} + \sqrt{\left(1 - \frac{E_{y0}^2}{E_{x0}^2}\right)^2 + 4\frac{E_{y0}^2}{E_{x0}^2} \cos^2(\epsilon)}}} \quad (119)$$

An expression for the angle of the major axis of the ellipse was found in 115. Expressed in terms of the field ratio parameter E_{y0}/E_{x0} and the relative phase ϵ , the expression for ψ becomes

$$\psi = \frac{1}{2} \tan^{-1} \left(\frac{\frac{E_{y0}}{E_{x0}} \cos(\epsilon)}{1 - \frac{E_{y0}^2}{E_{x0}^2}} \right) \quad (120)$$

The \tan^{-1} function is multivalued. We restrict its range to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ where it is single valued, and use other criteria to add or subtract π to get to the next branch of the \tan^{-1} function in order to determine ψ over its entire range. Because of its symmetry, an ellipse angle ψ greater than $\frac{\pi}{2}$ is equivalent to a negative angle $\psi - \pi$. Because of this, the range of ψ is restricted to be $[-\frac{\pi}{2}, \frac{\pi}{2}]$. When $E_{y0} \leq E_{x0}$ and $0 \leq |\epsilon| \leq \frac{\pi}{2}$, the \tan^{-1} function is in the range $[0, \frac{\pi}{2}]$, and ψ is in the range $[0, \frac{\pi}{4}]$. When $E_{y0} \leq E_{x0}$ and $\frac{\pi}{2} \leq |\epsilon| \leq \pi$, the \tan^{-1} function is in the range $[-\frac{\pi}{2}, 0]$, and ψ is in

the range $[-\frac{\pi}{4}, 0]$. When $E_{y0} \geq E_{x0}$ and $0 \leq |\epsilon| \leq \frac{\pi}{2}$, the \tan^{-1} function is in the range $[-\frac{\pi}{2}, 0]$. We add π to get to the next branch of the \tan^{-1} function making the range $[\frac{\pi}{2}, \pi]$. This makes ψ in the range $[\frac{\pi}{4}, \frac{\pi}{2}]$. When $E_{y0} \geq E_{x0}$ and $\frac{\pi}{2} \leq |\epsilon| \leq \pi$, the \tan^{-1} function is in the range $[0, \frac{\pi}{2}]$. We subtract π to get to the next branch of the \tan^{-1} function making the range $[-\pi, -\frac{\pi}{2}]$. This makes ψ in the range $[-\frac{\pi}{2}, -\frac{\pi}{4}]$.

The rotation direction around the E_x, E_y plane is determined by ϵ , as we have already seen. Positive ϵ produces an electric field which rotates clockwise. Negative ϵ produces an electric field which rotates counterclockwise. Positive ϵ produces a right handed wave. Negative ϵ produces a left handed wave.

The electric field parameters can be expressed in terms of the polarization ellipse parameters using equations 112 to 114. From equations 112 and 113, we have

$$\frac{E_{y0}}{E_{x0}} = \frac{\sqrt{\frac{1}{2}[(a^2 + b^2) - (a^2 - b^2) \cos(2\psi)]}}{\sqrt{\frac{1}{2}[(a^2 + b^2) + (a^2 - b^2) \cos(2\psi)]}} \quad (121)$$

Expressed in terms of the ratio of the minor to major ellipse axis lengths and the rotation angle of the ellipse, this becomes

$$\frac{E_{y0}}{E_{x0}} = \sqrt{\frac{(1 + \frac{b^2}{a^2}) - (1 - \frac{b^2}{a^2}) \cos(2\psi)}{(1 + \frac{b^2}{a^2}) + (1 - \frac{b^2}{a^2}) \cos(2\psi)}} \quad (122)$$

From 114, we find ϵ in terms of the polarization ellipse parameters.

$$\epsilon = \cos^{-1} \left(\frac{\sin(2\psi)}{|\sin(2\psi)|} \sqrt{\frac{(1 - \frac{b^2}{a^2})^2}{1 + \frac{b^4}{a^4} + \frac{b^2}{a^2} (\tan^2 \psi + \cot^2 \psi)}} \right) \quad (123)$$

In order to make the \cos^{-1} function single valued, its range is restricted to $[0, \pi]$ and it can not distinguish between positive and negative ϵ . We choose the sign of ϵ based on the rotation direction around the ellipse. Clockwise rotation (right hand helicity) gives positive ϵ and counterclockwise rotation (left hand helicity) gives negative ϵ .

The radiation wavelength, or K value from the undulator, is not specified by the polarization ellipse. It must be measured separately. Note, however, that $a^2 + b^2 = E_{x0}^2 + E_{y0}^2 = I$, which we have seen is a function of K . In order to keep K constant while changing the parameters of the polarization ellipse, one must keep $a^2 + b^2$ constant.

6 Stokes Parameters

6.1 Equations Of The Stokes Parameters

The Stokes parameters are defined in terms of the electric field parameters⁷.

$$S_0 = E_{x0}^2 + E_{y0}^2 \quad (124)$$

$$S_1 = E_{x0}^2 - E_{y0}^2 \quad (125)$$

$$S_2 = 2E_{x0}E_{y0} \cos(\epsilon) \quad (126)$$

$$S_3 = 2E_{x0}E_{y0} \sin(\epsilon) \quad (127)$$

The Stokes parameters are related by

$$S_0^2 = S_1^2 + S_2^2 + S_3^2 \quad (128)$$

⁷E. Hecht, *Optics*, 3rd edition, Addison Wesley Longman, Reading, Massachusetts, 1998.

The reverse transformation is easily found, the electric field parameters in terms of the Stokes parameters are given by

$$E_{x0} = \sqrt{\frac{S_0 + S_1}{2}} \quad (129)$$

$$E_{y0} = \sqrt{\frac{S_0 - S_1}{2}} \quad (130)$$

$$\tan(\epsilon) = \frac{S_3}{S_2} \quad (131)$$

6.2 Relation Between The Stokes Parameters And The Electric Field Parameters

It is useful to normalize the Stokes parameters to the intensity S_0 . The parameters we use for defining the polarization mode of the undulator are the ratios S_1/S_0 , S_2/S_0 , and S_3/S_0 . These ratios are not independent and are related by

$$\left(\frac{S_1}{S_0}\right)^2 + \left(\frac{S_2}{S_0}\right)^2 + \left(\frac{S_3}{S_0}\right)^2 = 1 \quad (132)$$

All three ratios are required, however, in order to determine the sign of ϵ . The Stokes parameter ratios do not give the wavelength of the radiation, which must be measured separately.

The ratios of the Stokes parameters can be expressed in terms of the electric field parameter ratio E_{y0}/E_{x0} and the relative phase ϵ . The ratio S_1/S_0 is given by

$$\frac{S_1}{S_0} = \frac{1 - \left(\frac{E_{y0}}{E_{x0}}\right)^2}{1 + \left(\frac{E_{y0}}{E_{x0}}\right)^2} \quad (133)$$

The ratio S_2/S_0 is given by

$$\frac{S_2}{S_0} = \frac{2\frac{E_{y0}}{E_{x0}}}{1 + \left(\frac{E_{y0}}{E_{x0}}\right)^2} \cos(\epsilon) \quad (134)$$

The ratio S_3/S_0 is given by

$$\frac{S_3}{S_0} = \frac{2\frac{E_{y0}}{E_{x0}}}{1 + \left(\frac{E_{y0}}{E_{x0}}\right)^2} \sin(\epsilon) \quad (135)$$

The electric field parameters can be expressed in terms of the Stokes parameter ratios. The field parameter ratio E_{y0}/E_{x0} is given by

$$\frac{E_{y0}}{E_{x0}} = \sqrt{\frac{1 - \frac{S_1}{S_0}}{1 + \frac{S_1}{S_0}}} \quad (136)$$

The relative phase difference ϵ is given by

$$\epsilon = (\tan 2)^{-1} \left(\frac{S_3}{S_0}, \frac{S_2}{S_0} \right) \quad (137)$$

where $(\tan 2)^{-1}$ is meant to represent the four quadrant inverse tangent. The four quadrant inverse tangent gives the sign of ϵ , so a separate determination of the helicity is not required, unlike the case for the polarization ellipse.

As noted previously, the radiation wavelength, or equivalently the K value of the undulator, is not determined by the Stokes parameters in an easily accessible way. S_0 is a function of K , but it is much easier to measure the wavelength instead in order to determine K .

7 Relations Between The Stokes Parameters And The Polarization Ellipse

Using equations 115 to 117 and 124 to 127, we can easily read off the polarization ellipse parameters in terms of the Stokes parameters.

$$\tan(2\psi) = \frac{S_2}{S_1} \quad (138)$$

$$a^2 = \frac{1}{2} \left(S_0 + \sqrt{S_1^2 + S_2^2} \right) \quad (139)$$

$$b^2 = \frac{1}{2} \left(S_0 - \sqrt{S_1^2 + S_2^2} \right) \quad (140)$$

These can be solved to give

$$\psi = \frac{1}{2} \tan^{-1} \left(\frac{S_2}{S_1} \right) \quad (141)$$

$$a = \sqrt{\frac{1}{2} \left(S_0 + \sqrt{S_1^2 + S_2^2} \right)} \quad (142)$$

$$b = \sqrt{\frac{1}{2} \left(S_0 - \sqrt{S_1^2 + S_2^2} \right)} \quad (143)$$

As noted previously, we restrict the range of the \tan^{-1} function to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ where it is single valued and use other criteria to add or subtract π to get to the next branch in order to determine ψ over its entire range. When $S_1 \geq 0$ and $S_2 \geq 0$, the \tan^{-1} function is in the range $[0, \frac{\pi}{2}]$ and ψ is in the range $[0, \frac{\pi}{4}]$. When $S_1 \geq 0$ and $S_2 \leq 0$, the \tan^{-1} function is in the range $[-\frac{\pi}{2}, 0]$ and ψ is in the range $[-\frac{\pi}{4}, 0]$. When $S_1 \leq 0$ and $S_2 \geq 0$, the \tan^{-1} function is in the range $[-\frac{\pi}{2}, 0]$. We add π to get to the next branch of the \tan^{-1} function making the range $[\frac{\pi}{2}, \pi]$. This makes ψ in the range $[\frac{\pi}{4}, \frac{\pi}{2}]$. When $S_1 \leq 0$ and $S_2 \leq 0$, the \tan^{-1} function is in the range $[0, \frac{\pi}{2}]$. We subtract π to get to the next branch of the \tan^{-1} function making the range $[-\pi, -\frac{\pi}{2}]$. This makes ψ in the range $[-\frac{\pi}{2}, -\frac{\pi}{4}]$.

The inverse transformation is given by expressing the Stokes parameters in terms of the electric field parameters, and then expressing the electric field parameters in terms of the polarization ellipse parameters. Using equations 112 to 114 and 124 to 127, we get

$$S_0 = a^2 + b^2 \quad (144)$$

$$S_1 = (a^2 - b^2) \cos(2\psi) \quad (145)$$

$$S_2 = (a^2 - b^2) \sin(2\psi) \quad (146)$$

$$S_3 = 2ab \quad (147)$$

8 Laboratory Frame

The polarization parameters are measured in the laboratory frame, rather than in the rotated frame of the undulator. A transformation is required to go between the laboratory frame and the undulator frame.

The polarization ellipse in the laboratory frame has the same ratio of minor to major axis. The rotation angle of the ellipse in the laboratory frame is just $\pi/4$ plus the rotation angle in the undulator frame, since the undulator frame is rotated by $\pi/4$ in the laboratory frame.

The Stokes parameters change since the field components change in the laboratory frame. The transformation of the field components is

$$E_{xL} = \frac{1}{\sqrt{2}}(E_x - E_y) \quad (148)$$

$$E_{yL} = \frac{1}{\sqrt{2}}(E_x + E_y) \quad (149)$$

where the subscript L indicates the laboratory frame. In complex notation, the field components in the laboratory at a fixed z -location are

$$E_{xL} = \frac{1}{\sqrt{2}}(E_{x0} - E_{y0}e^{i\epsilon})e^{i\omega_r t} \quad (150)$$

$$E_{yL} = \frac{1}{\sqrt{2}}(E_{x0} + E_{y0}e^{i\epsilon})e^{i\omega_r t} \quad (151)$$

When the electric field is expressed in complex form, the Stokes parameters are given by

$$S_0 = E_{x0}E_{x0}^* + E_{y0}E_{y0}^* \quad (152)$$

$$S_1 = E_{x0}E_{x0}^* - E_{y0}E_{y0}^* \quad (153)$$

$$S_2 = E_{x0}E_{y0}^* + E_{x0}^*E_{y0} \quad (154)$$

$$S_3 = i(E_{x0}E_{y0}^* - E_{x0}^*E_{y0}) \quad (155)$$

Applying these formulas to the fields in the laboratory frame, we find the Stokes parameters to be

$$S_{0L} = \frac{1}{2}[(E_{x0} - E_{y0}e^{i\epsilon})(E_{x0} - E_{y0}e^{-i\epsilon}) + (E_{x0} + E_{y0}e^{i\epsilon})(E_{x0} + E_{y0}e^{-i\epsilon})] \quad (156)$$

$$= E_{x0}^2 + E_{y0}^2 = S_0 \quad (157)$$

$$S_{1L} = \frac{1}{2}[(E_{x0} - E_{y0}e^{i\epsilon})(E_{x0} - E_{y0}e^{-i\epsilon}) - (E_{x0} + E_{y0}e^{i\epsilon})(E_{x0} + E_{y0}e^{-i\epsilon})] \quad (158)$$

$$= -2E_{x0}E_{y0}\cos(\epsilon) = -S_2 \quad (159)$$

$$S_{2L} = \frac{1}{2}[(E_{x0} - E_{y0}e^{i\epsilon})(E_{x0} + E_{y0}e^{-i\epsilon}) + (E_{x0} - E_{y0}e^{-i\epsilon})(E_{x0} + E_{y0}e^{i\epsilon})] \quad (160)$$

$$= E_{x0}^2 - E_{y0}^2 = S_1 \quad (161)$$

$$S_{3L} = i\frac{1}{2}[(E_{x0} - E_{y0}e^{i\epsilon})(E_{x0} + E_{y0}e^{-i\epsilon}) - (E_{x0} - E_{y0}e^{-i\epsilon})(E_{x0} + E_{y0}e^{i\epsilon})] \quad (162)$$

$$= 2E_{x0}E_{y0}\sin(\epsilon) = S_3 \quad (163)$$

In summary, the transformation of the Stokes parameters between the laboratory frame and the undulator frame is

$$S_{0L} = S_0 \quad (164)$$

$$S_{1L} = -S_2 \quad (165)$$

$$S_{2L} = S_1 \quad (166)$$

$$S_{3L} = S_3 \quad (167)$$

9 Conclusion

This note provided the path between the row positions of the undulator and the desired polarization mode. It also provided the reverse path between the desired polarization mode and the row positions. The individual transformations from row positions to magnetic field, from magnetic field to electric field, and from electric field to polarization parameters were presented. The reverse transformations were also given. These transformations can be applied sequentially to go from row positions to polarization parameters, or from polarization parameters to row positions.

10 Summary Of Results

A summary of the results derived above is presented here. First we show the results to go from given row positions to the resulting polarization mode. Then we shown the results to go from a given polarization mode to the required row positions. Restrictions on the ranges of quantities have been detailed in the text.

10.1 Row Positions To Polarization Mode

10.1.1 Row Positions To Magnetic Field

$$\frac{K}{K_0} = \frac{1}{\sqrt{2}} \sqrt{\cos^2 \left(k_u \frac{z_{01} - z_{03}}{2} \right) + \cos^2 \left(k_u \frac{z_{02} - z_{04}}{2} \right)}$$

$$\frac{B_{y0}}{B_{x0}} = \frac{\cos \left(k_u \frac{z_{02} - z_{04}}{2} \right)}{\cos \left(k_u \frac{z_{01} - z_{03}}{2} \right)}$$

$$\delta = \frac{k_u}{2} [(z_{01} + z_{03}) - (z_{02} + z_{04})]$$

10.1.2 Magnetic Field To Electric Field

$$\lambda_r = \frac{\lambda_u}{2\gamma^2} \left[1 + \frac{1}{2} \left(\frac{K}{K_0} \right)^2 K_0^2 \right]$$

$$\frac{E_{y0}}{E_{x0}} = \frac{B_{x0}}{B_{y0}}$$

$$\epsilon = -\delta + \pi$$

10.1.3 Electric Field To Polarization Ellipse

$$\frac{b}{a} = \frac{\sqrt{1 + \frac{E_{y0}^2}{E_{x0}^2} - \sqrt{\left(1 - \frac{E_{y0}^2}{E_{x0}^2}\right)^2 + 4 \frac{E_{y0}^2}{E_{x0}^2} \cos^2(\epsilon)}}}{\sqrt{1 + \frac{E_{y0}^2}{E_{x0}^2} + \sqrt{\left(1 - \frac{E_{y0}^2}{E_{x0}^2}\right)^2 + 4 \frac{E_{y0}^2}{E_{x0}^2} \cos^2(\epsilon)}}$$

$$\psi = \frac{1}{2} \tan^{-1} \left(2 \frac{\frac{E_{y0}}{E_{x0}} \cos(\epsilon)}{1 - \frac{E_{y0}^2}{E_{x0}^2}} \right)$$

Positive ϵ produces an electric field which rotates clockwise around the ellipse. Negative ϵ produces an electric field which rotates counterclockwise.

10.1.4 Electric Field To Stokes Parameters

$$\frac{S_1}{S_0} = \frac{1 - \left(\frac{E_{y0}}{E_{x0}}\right)^2}{1 + \left(\frac{E_{y0}}{E_{x0}}\right)^2}$$

$$\frac{S_2}{S_0} = \frac{2\frac{E_{y0}}{E_{x0}}}{1 + \left(\frac{E_{y0}}{E_{x0}}\right)^2} \cos(\epsilon)$$

$$\frac{S_3}{S_0} = \frac{2\frac{E_{y0}}{E_{x0}}}{1 + \left(\frac{E_{y0}}{E_{x0}}\right)^2} \sin(\epsilon)$$

10.2 Polarization Mode To Row Positions

10.2.1 Stokes Parameters To Electric Field

$$\frac{E_{y0}}{E_{x0}} = \sqrt{\frac{1 - \frac{S_1}{S_0}}{1 + \frac{S_1}{S_0}}}$$

$$\epsilon = (\tan 2)^{-1} \left(\frac{S_3}{S_0}, \frac{S_2}{S_0} \right)$$

where $(\tan 2)^{-1}$ is meant to represent the four quadrant inverse tangent. The third parameter, the wavelength λ_r , must be measured independently.

10.2.2 Polarization Ellipse To Electric Field

$$\frac{E_{y0}}{E_{x0}} = \sqrt{\frac{(1 + \frac{b^2}{a^2}) - (1 - \frac{b^2}{a^2}) \cos(2\psi)}{(1 + \frac{b^2}{a^2}) + (1 - \frac{b^2}{a^2}) \cos(2\psi)}}$$

$$\epsilon = \cos^{-1} \left(\frac{\sin(2\psi)}{|\sin(2\psi)|} \sqrt{\frac{(1 - \frac{b^2}{a^2})^2}{1 + \frac{b^4}{a^4} + \frac{b^2}{a^2} (\tan^2 \psi + \cot^2 \psi)}} \right)$$

The sign of ϵ is based on the rotation direction around the ellipse. Clockwise rotation (right hand helicity) gives positive ϵ and counterclockwise rotation (left hand helicity) gives negative ϵ . The third parameter, the wavelength λ_r , must be measured independently.

10.2.3 Electric Field To Magnetic Field

$$\frac{K}{K_0} = \sqrt{\frac{2}{K_0^2} \left(\lambda_r \frac{2\gamma^2}{\lambda_u} - 1 \right)}$$

$$\frac{B_{y0}}{B_{x0}} = \frac{E_{x0}}{E_{y0}}$$

$$\delta = -\epsilon + \pi$$

10.2.4 Magnetic Field To Row Positions

$$\begin{aligned}
 z_{01} &= \frac{1}{2} \left[\frac{\delta}{k_u} + \frac{2}{k_u} \cos^{-1} \left(\sqrt{\frac{2 \left(\frac{K}{K_0}\right)^2}{1 + \left(\frac{B_{y0}}{B_{x0}}\right)^2}} \right) \right] \\
 z_{02} &= \frac{1}{2} \left[-\frac{\delta}{k_u} + \frac{2}{k_u} \cos^{-1} \left(\sqrt{\frac{2 \left(\frac{K}{K_0}\right)^2 \left(\frac{B_{y0}}{B_{x0}}\right)^2}{1 + \left(\frac{B_{y0}}{B_{x0}}\right)^2}} \right) \right] \\
 z_{03} &= \frac{1}{2} \left[\frac{\delta}{k_u} - \frac{2}{k_u} \cos^{-1} \left(\sqrt{\frac{2 \left(\frac{K}{K_0}\right)^2}{1 + \left(\frac{B_{y0}}{B_{x0}}\right)^2}} \right) \right] \\
 z_{04} &= \frac{1}{2} \left[-\frac{\delta}{k_u} - \frac{2}{k_u} \cos^{-1} \left(\sqrt{\frac{2 \left(\frac{K}{K_0}\right)^2 \left(\frac{B_{y0}}{B_{x0}}\right)^2}{1 + \left(\frac{B_{y0}}{B_{x0}}\right)^2}} \right) \right]
 \end{aligned}$$