Hall Probe Array Measurements Of The Delta Undulator

Zachary Wolf
SLAC

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Abstract

This note discusses methods to obtain required undulator parameter values from the Hall probe array measurements of the Delta undulator. In particular, the probe position relative to the magnetic center is found in both linear and circular polarization modes. This is required for the undulator ficucialization. Parameters that describe the field variation with transverse position are also determined. This information lets us calculate the fields on the beam axis given the fields at the measurement location. The measurement accuracy required to make these determinations is discussed.

1 Introduction[1]

LCLS technical note LCLS-TN-13-4[2] presented a measurement plan for the Delta undulator. This note is a more detailed analysis of the Hall probe array measurements discussed in the plan. Two Hall probes offset in one direction make up the array. Each probe measures all three field components. In this note, we discuss how to use the probe array measurements to extract as much information as possible about the fields. In particular, we must determine the probe position relative to the magnetic center in order to fiducialize the undulator. We also must determine the parameters which describe how the field varies in the transverse directions. This information will let us calculate the fields on the beam axis given the fields at the measurement location.

This note provides a method to calculate field parameters given certain assumptions which will be presented. In practice, measurement errors and error fields from the magnets may make some of the calculations detailed below unusable since these unknown quantities are not accounted for. These calculations provide a path to begin analyzing the undulator measurements. Consistency of the results in repeated measurements and in measurements of the different undulator modes will play a large part in determining whether unaccounted for errors are influencing the results.

2 Overview

As noted in LCLS-TN-13-4, the Hall probe array follows a curved path through the undulator and measures the three field components on this path. This is illustrated in figure 1. The path of the probes is found relative to a straight line by a laser system that measures transverse position changes. The probes are found relative to fiducials at the two ends of the undulator using high gradient fiducialization magnets. The two points define a line and the probe position is calculated relative to this line using the laser measurements. We take this line to be the axis of the coordinate system. y_1 is the position of probe 1 relative to the coordinate system axis in the figure.

[1] Work supported in part by the DOE Contract DE-AC02-76SF00515. This work was performed in support of the LCLS project at SLAC.

[2] Z. Wolf, "A Magnetic Measurement Plan For The Delta Undulator", LCLS-TN-13-4, March, 2013.

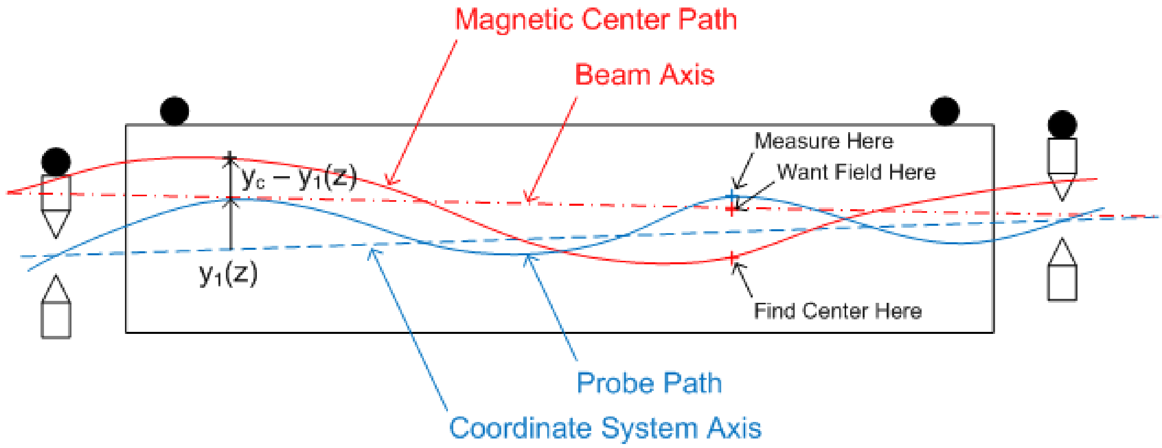

Figure 1: The field is measured on the probe path. The probe path is know relative to the coordinate system axis which is defined by the probe positions at the two ends of the undulator. The magnetic center position is calculated from the measurements. A linear fit to the magnetic centers defines the beam axis. The measured fields and the functional form of the fields allows the fields to be calculated on the beam axis.

The field measurements allow us to calculate the position of the magnetic center relative to the probe. This is denoted by $y_c - y_1$ in the figure. These calculations are the primary subject of this note. To make these calculations, the functional form the the fields must be specified. All parameters in the functional form must be calculated from the measurements. The way the functional form is determined is that Maxwell's equations are used to calculate the form of the terms in a series expansion of the field. We assume that near the magnetic center, there is a largest term at the fundamental longitudinal frequency and that it dominates the series expansion. This term is what we use for the functional form of the field. An expansion in the transverse coordinates is made assuming the transverse coordinates are small enough that a second order expansion is adequate. We use the form of the transverse expansion for the form of the fields. Basically, we are doing a second order expansion of the transverse coordinates of the fields, and we use the Maxwell's equation solution to guide the form of the quadratic. As long as we are close enough to the magnetic center for the second order expansion to be accurate, this technique should provide an adequate parameterization of the fields. The coefficients in the quadratic expansion are parameters that must be determined from the measurements.

Once we know the position of the magnetic center and the parameters in the functional form of the fields, we can calculate the fields at an arbitrary point. We fit the magnetic centers with a line to determine the beam axis. We fiducialize to this axis. We then calculate the fields on this line to determine the undulator characteristics.

In this note, the probe position relative to the magnetic center is found for horizontal and vertical linear, and right and left circular polarization modes. In principle, the undulator can be fiducialized for each mode, even if the magnetic center is changing. In practice, only one fiducialization will be used. The constancy of the magnetic center position will be checked.

In order for a second order expansion of the fields in the transverse coordinates to be accurate, the magnetic center must follow a straight line down the center of the magnet bore at the 100 μm level, and the measurement probes must stay near the magnet bore center at the 100 μm level. Combined, the measurement probes must stay within about 200 μm of the magnetic center. This requirement is motivated by the fact that the expansion is in terms of a quantity kr, where k is k_x

or k_y, the parameter describing the behavior of the field in the transverse direction, and r is either x or y, the transverse distance from the magnetic center. Typically $k \simeq 200$ 1/m. If we wish to keep $kr < 10^{-1}$, we must keep $r < 5 \times 10^{-4}$ m. Keeping the probes closer to the magnetic center than this, say at the 200 μm level, improves accuracy of the transverse field expansion and is highly desirable.

3 Hall Probe Measurements

3.1 Probe Array

The Hall probe array contains two probes offset in the B_y sensor direction. The array can be rotated so the field can be sampled at two points offset in any direction. The probe outputs have a rotation accounted for in software, so they always give fields in the original xyz coordinate system. We only use two cases, one with the offset direction along the y-axis, and the other with the offset direction along the x-axis.

The Hall probe array has the probe elements aligned in x and y, but distributed in z. In order to minimize the error coming from knowing the precise z-position of the measurement, we only use the field peaks in the calculations of the undulator parameters since their values are insensitive to the exact z-position of the measurement. Another advantage of using the field peak values is that they are fitted so that errors from measurement noise are reduced.

3.2 Measurement Errors

In order to discuss errors, we must first consider the magnitudes of the fields, the magnitudes of the errors, and the approximations used in the calculations. From the undulator simulations, k_x and k_y are both of order 2×10^2 1/m. From the construction of the undulator and the construction of the probe, we expect the distance from the probe element to the magnetic center to be less then 5×10^{-4} m. Using these numbers for vertical field linear polarization mode, for instance, we expect the peak fields at the probe positions to be in the ratio $B_x/B_y \sim k^2xy < 10^{-2}$, and $B_z/B_y \sim ky < 10^{-1}$. In the calculations, we expand the fields in $k_x x$ and $k_y y$, both are of order 10^{-1}, or smaller. Fields like B_x in the above example are second order in the expansion in transverse coordinates, and B_z is first order. The probes will measure close enough to the magnetic center so that the first and second order expansion in $k_x x$ and $k_y y$ will be good approximations of the fields. Note that this estimate is an upper limit on $k_x x$ and $k_y y$. These quantities may be much smaller depending on x and y, and this will be important when we discuss measurement errors.

Probe package angular misalignments are expected to be of order .01 rad, and they must be included in the calculations in several cases. First of all, they can mix the strong component of a field into a weak component measurement through terms first order in the misalignment angle. Second, when differences in the strong component of the field are used in a calculation, second order terms in the misalignment angle, cosine errors, can be significant compared to the difference.

Roll errors of the probe elements linearly add B_y to the B_x signal in linearly polarized vertical field mode, as an example. This problem is most severe if the probes are well centered in x, since the B_x field will be small and the error signal may be large. Note that in linear mode, the field peaks of B_x occur at the same locations as the field peaks of B_y. This makes it very difficult to separate a B_y signal mixed into B_x. Because it is difficult to be sure the B_x signal is not contaminated by probe roll or individual Hall element roll, we must use care when we use B_x in any of the parameter determinations. This problem does not occur for B_z since the field peaks of B_z are at the zeros of B_x and B_y in the linear modes. Probe pitch errors would potentially mix B_y and B_z, but since we are using the field peaks, and the peaks of B_z occur at the zeros of B_y, this problem does not arise.

Hall element roll changes the main field component measurement by the cosine of the element roll angle. The second order expansion of the cosine times the large main field can give errors of similar magnitude to the terms of interest in the calculation. Such effects will be discussed below.

Both the first and second order effects of the angle errors of the Hall elements have a large impact on the measurements. The angle errors of the Hall elements can be measured and corrected, however. The effect of the Hall element angle errors and the required accuracy of the angle measurements will be discussed below.

Overall probe roll is difficult to measure and control. We will make an effort to minimize the effect of overall probe roll in the analysis below. We will discuss the limits on probe roll.

In addition to angle errors, the Hall probe array has position errors in its construction which must be taken into account. Even though the main offset in the two probes is in the y-direction, the two probes will also have a slight offset in the B_x sensor direction. This error is included in the discussion below so its effect can be understood.

3.3 Hall Element Angle Error Corrections

Each Hall element has position errors and angular alignment errors. These errors will be determined using fiducialization magnets and dipole calibration magnets. The position errors will be dealt with explicitly in the calculations in the following sections.

Hall element angle errors must be measured in a calibration procedure. They can not be extracted from the analysis of the measurements. The measurements will be corrected to account for the angle errors. This will be done prior to the analysis.

These angles are only the angular errors in the probe construction. Overall roll, pitch, and yaw of the probe is not included. As will be shown, we require that roll, pitch, and yaw be on the order of 2 mrad so that they can be neglected. Construction errors in the probe, however, are specified to be on the order of 25 mrad and must be corrected. In the calibration, one element is taken as the zero reference, and that element is used to set the overall probe angles.

3.3.1 Probes Offset In The y-Direction

Consider a B_x Hall element. Ideally it only measures B_x, but roll, pitch, and yaw angle errors, Θ_{xr}, Θ_{xp}, and Θ_{xy}, respectively, mix the signals from the field components. We assume the errors are small. The element measures

$$S_x = \cos(\Theta_{xr})\cos(\Theta_{xy})B_x + \sin(\Theta_{xr})B_y + \sin(\Theta_{xy})B_z \tag{1}$$

Similarly a B_y Hall element ideally only measures B_y, but roll, pitch, and yaw angle errors, Θ_{yr}, Θ_{yp}, and Θ_{yy}, respectively, mix the signals from the field components, so it measures

$$S_y = -\sin(\Theta_{yr})B_x + \cos(\Theta_{yr})\cos(\Theta_{yp})B_y - \sin(\Theta_{yp})B_z \tag{2}$$

A B_z Hall element ideally only measures B_z, but roll, pitch, and yaw angle errors, Θ_{zr}, Θ_{zp}, and Θ_{zy}, respectively, mix the signals from the field components, so it measures

$$S_z = -\sin(\Theta_{zy})B_x + \sin(\Theta_{zp})B_y + \cos(\Theta_{zy})\cos(\Theta_{zp})B_z \tag{3}$$

These equations can be put in matrix form

$$\begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \begin{pmatrix} \cos(\Theta_{xr})\cos(\Theta_{xy}) & \sin(\Theta_{xr}) & \sin(\Theta_{xy}) \\ -\sin(\Theta_{yr}) & \cos(\Theta_{yr})\cos(\Theta_{yp}) & -\sin(\Theta_{yp}) \\ -\sin(\Theta_{zy}) & \sin(\Theta_{zp}) & \cos(\Theta_{zy})\cos(\Theta_{zp}) \end{pmatrix} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} \tag{4}$$

Representing this equation as $S = AB$, we measure S, we calibrate to determine A, and we wish to know the actual fields B. We do this by inverting the A matrix and calculating $B = A^{-1}S$. We perform this correction before we start the analysis of the fields.

3.3.2 Probes Offset In The x-Direction

When the probe assembly is rotated so the probe offset is in the x-direction, the signals from the Hall elements depend on the fields differently than above. The angle errors for each element are the same, however. Consider a B_x Hall element. When the probe is rotated, it primarily measures $-B_y$. Roll, pitch, and yaw angle errors, Θ_{xr}, Θ_{xp}, and Θ_{xy}, respectively, mix the signals from the field components, so it measures

$$S_x = \sin(\Theta_{xr})B_x - \cos(\Theta_{xr})\cos(\Theta_{xp})B_y + \sin(\Theta_{xp})B_z \tag{5}$$

Similarly a B_y Hall element ideally only measures B_x, but roll, pitch, and yaw angle errors, Θ_{yr}, Θ_{yp}, and Θ_{yy}, respectively, mix the signals from the field components, so it measures

$$S_y = \cos(\Theta_{yr})\cos(\Theta_{yy})B_x + \sin(\Theta_{yr})B_y + \sin(\Theta_{yy})B_z \tag{6}$$

A B_z Hall element ideally only measures B_z, but roll, pitch, and yaw angle errors, Θ_{zr}, Θ_{zp}, and Θ_{zy}, respectively, mix the signals from the field components, so it measures

$$S_z = -\sin(\Theta_{zy})B_x + \sin(\Theta_{zp})B_y + \cos(\Theta_{zy})\cos(\Theta_{zp})B_z \tag{7}$$

These equations can be put in matrix form

$$\begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \begin{pmatrix} \sin(\Theta_{xr}) & -\cos(\Theta_{xr})\cos(\Theta_{xp}) & \sin(\Theta_{xp}) \\ \cos(\Theta_{yr})\cos(\Theta_{yy}) & +\sin(\Theta_{yr}) & \sin(\Theta_{yy}) \\ -\sin(\Theta_{zy}) & \sin(\Theta_{zp}) & \cos(\Theta_{zy})\cos(\Theta_{zp}) \end{pmatrix} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} \tag{8}$$

Representing this equation as $S = AB$, we measure S, we calibrate to determine A, and we wish to know the actual fields B. We do this by inverting the A matrix and calculating $B = A^{-1}S$. We perform this correction before we start the analysis of the fields.

3.4 Probe Transfer Function Calibration Accuracy Requirements

Since B_x and B_y are used in calculations involving the second order field expansion, the errors in the fields from the Hall element calibrations must be significantly less than the second order terms in the expansion. We write this as

$$\delta B \ll B_0 \left(kr\right)^2 \tag{9}$$

where r represents x or y, and k represents k_x or k_y. B_0 represents the peak field amplitude. Since k might be as small as 1×10^2 1/m, and r might be as small as 1×10^{-4} m, giving $kr = 1 \times 10^{-2}$, we require the B_x and B_y calibration to be accurate to

$$\frac{\delta B}{B_0} \ll \left(1 \times 10^{-2}\right)^2 \tag{10}$$

or

$$\frac{\delta B}{B_0} \ll 10^{-4} \tag{11}$$

Since B_z is used in calculations involving the first order field expansion, the errors in B_z from its calibration must be less than the first order term in the expansion. We write this as

$$\delta B \ll B_0 kr \tag{12}$$

using the notation given above. Using the same estimates for k and r, we require the B_z calibration to be accurate to

$$\frac{\delta B}{B_0} \ll 10^{-2} \tag{13}$$

4 Measurements In The Different Undulator Modes

In this section, we study the expressions for the fields in the undulator so that we can find measurements which determine the parameters that characterize the fields. We wish to determine the position of the probes relative to the magnetic center. We also wish to determine the parameters k_x and k_y which characterize the transverse behavior of the fields.

In the following sections we refer to expansions of the fields. These are first and second order expansions in the transverse coordinates. They are expansions in the small quantities $k_x x$ and $k_y y$.

4.1 Linear Polarization Vertical Field

The equations describing the fields were given in LCLS-TN-13-4. We use these as our starting point. We first consider the case where the probes are offset along the y-axis, and then consider the case where the probes are offset along the x-axis. Note that when we rotate the probes, their position shifts relative to the magnetic center. This is illustrated in figure 2. We must keep track of the probe orientation when we give the probe position relative to the magnetic center.

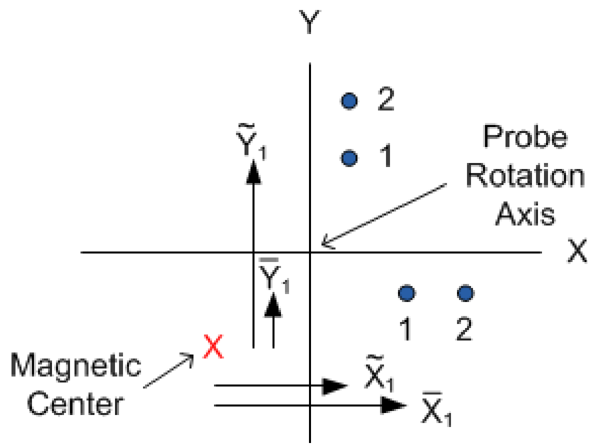

Figure 2: The probe element positions shift relative to the magnetic center as the probe is rotated.

4.1.1 Probes Offset In The y-Direction

Suppose the probes are offset along the y-axis. Let $\widetilde{x} = x - x_c$, where x is the x-position where the field is determined, and x_c is the x-position of the magnetic center. Similarly, $\widetilde{y} = y - y_c$, where y is the y-position where the field is determined, and y_c is the y-position of the magnetic center.

Consider the linear polarization vertical field mode. The fields are

$$B_x = \phi_0 k_x \sinh(k_x \widetilde{x}) \sinh(k_y \widetilde{y}) \cos(k_u z) \quad (14)$$
$$B_y = \phi_0 k_y \cosh(k_x \widetilde{x}) \cosh(k_y \widetilde{y}) \cos(k_u z) \quad (15)$$
$$B_z = -\phi_0 k_u \cosh(k_x \widetilde{x}) \sinh(k_y \widetilde{y}) \sin(k_u z) \quad (16)$$

Expanding to first order in $k_x \widetilde{x}$ and $k_y \widetilde{y}$, the fields become

$$B_x = 0 \quad (17)$$
$$B_y = \phi_0 k_y \cos(k_u z) \quad (18)$$
$$B_z = -\phi_0 k_u k_y \widetilde{y} \sin(k_u z) \quad (19)$$

Expanding to second order in $k_x\widetilde{x}$ and $k_y\widetilde{y}$, the fields become

$$
\begin{aligned}
B_x &= \phi_0 k_x k_x \widetilde{x} k_y \widetilde{y} \cos(k_u z) && (20)\\
B_y &= \phi_0 k_y \left[1 + \frac{1}{2}(k_x\widetilde{x})^2 + \frac{1}{2}(k_y\widetilde{y})^2\right] \cos(k_u z) && (21)\\
B_z &= -\phi_0 k_u k_y \widetilde{y} \sin(k_u z) && (22)
\end{aligned}
$$

We first use the linear expansions and only consider the field peak values to obtain

$$\frac{B_z}{B_y} = -k_u\widetilde{y} \tag{23}$$

Applying this to probe 1, we find

$$\boxed{\widetilde{y}_1 = y_1 - y_c = -\tfrac{1}{k_u}\left(\tfrac{B_z}{B_y}\right)_1} \tag{24}$$

So we determine the y-position of probe 1 relative to the magnetic center when the probe offset is vertical by using the ratio of the B_z and B_y peak fields from probe 1. $k_u = 2\pi/\lambda_u$ is known. In order to get the sign correct in this expression, we note that $B_z \sim \sin(k_u z)$ and $B_y \sim \cos(k_u z)$, so we must use the peak of B_z at the z-position $\lambda_u/4$ past the peak of B_y. Using B_z and B_y essentially eliminates the effect of probe pitch errors which mix B_y into B_z since B_y is zero at the peaks of B_z. Using the field peaks eliminates this mixing error from the calculation.

If we do a similar analysis for probe 2, we find

$$\widetilde{y}_2 = y_2 - y_c = -\frac{1}{k_u}\left(\frac{B_z}{B_y}\right)_2 \tag{25}$$

Noting that with the probes aligned so that the probes are offset in the y-direction, probe 2 is at $y_2 = y_1 + \Delta$, where Δ is the probe offset distance. Subtracting the probe 2 and probe 1 results, we find

$$\Delta = -\frac{1}{k_u}\left[\left(\frac{B_z}{B_y}\right)_2 - \left(\frac{B_z}{B_y}\right)_1\right] \tag{26}$$

or since $(B_y)_2 = (B_y)_1$ to first order,

$$\boxed{\Delta = -\tfrac{1}{k_u}\tfrac{(B_z)_2-(B_z)_1}{(B_y)_1}} \tag{27}$$

The offset distance between the probes is measured independently using fiducialization magnets, and the offset distance will be compared to the result calculated above using the fields.

To determine k_y, we use the second order expansion of B_y. Suppose we start with the probes offset in the y-direction. Let

$$
\begin{aligned}
\widetilde{y}_2 &= \widetilde{y}_1 + \Delta && (28)\\
\widetilde{x}_2 &= \widetilde{x}_1 + \delta && (29)
\end{aligned}
$$

Ideally, $\delta = 0$ so the probes have the same x-position, but we leave δ nonzero for now to determine its effect. Note that δ can also be measured using fiducialization magnets, so both Δ and δ are known. Returning to determining k_y using B_y, probe 2 measures peak fields with values

$$
\begin{aligned}
(B_y)_2 &= \phi_0 k_y \left[1 + \frac{1}{2}(k_x\widetilde{x}_2)^2 + \frac{1}{2}(k_y\widetilde{y}_2)^2\right] && (30)\\
&= \phi_0 k_y \left[1 + \frac{1}{2}k_x^2\left(\widetilde{x}_1^2 + 2\widetilde{x}_1\delta + \delta^2\right) + \frac{1}{2}k_y^2\left(\widetilde{y}_1^2 + 2\widetilde{y}_1\Delta + \Delta^2\right)\right] && (31)\\
&= (B_y)_1 + \phi_0 k_y \left[\frac{1}{2}k_x^2\left(2\widetilde{x}_1\delta + \delta^2\right) + \frac{1}{2}k_y^2\left(2\widetilde{y}_1\Delta + \Delta^2\right)\right] && (32)
\end{aligned}
$$

Working to second order, we find

$$\frac{(B_y)_2 - (B_y)_1}{(B_y)_1} = \left[\frac{1}{2}k_x^2\left(2\widetilde{x}_1\delta + \delta^2\right) + \frac{1}{2}k_y^2\left(2\widetilde{y}_1\Delta + \Delta^2\right)\right] \tag{33}$$

Solving for k_y^2, we find

$$\boxed{k_y^2 = \frac{(B_y)_2 - (B_y)_1}{(B_y)_1}\frac{1}{\left(\widetilde{y}_1\Delta + \frac{1}{2}\Delta^2\right)} - k_x^2\frac{\widetilde{x}_1\delta + \frac{1}{2}\delta^2}{\widetilde{y}_1\Delta + \frac{1}{2}\Delta^2}} \tag{34}$$

At this point, we do not know $\widetilde{x}_1$, but we expect $\widetilde{x}_1$ to be of the same order as $\widetilde{y}_1$. We expect $\delta \ll \Delta$. So, we expect

$$\frac{\widetilde{x}_1\delta + \frac{1}{2}\delta^2}{\widetilde{y}_1\Delta + \frac{1}{2}\Delta^2} \ll 1 \tag{35}$$

This lets us make an initial estimate of k_y^2.

$$k_y^2 = \frac{(B_y)_2 - (B_y)_1}{(B_y)_1}\frac{1}{\left(\widetilde{y}_1\Delta + \frac{1}{2}\Delta^2\right)} \tag{36}$$

Once we calculate $\widetilde{x}_1$, we will refine the value of k_y^2 using equation 34. The constraint on the k's gives us k_x^2.

$$k_x^2 = k_u^2 - k_y^2 \tag{37}$$

Consider the effect of various errors on the expression for k_y^2. Since B_x is second order, we can neglect the effect of it mixing into the second order expansion of B_y from probe roll. Since the peaks of B_y occur at the zeros of B_z, we can neglect the effect of B_z mixing with B_y from probe pitch. Hall element angle errors give cosine errors on B_y, but these are assumed to have been measured and corrected. The largest source of error is likely the error on $\widetilde{x}_1$ in equation 34. If we assume $\widetilde{x}_1$ is of the same order as $\widetilde{y}_1$, and we neglect this second term and use equation 36, the error on k_y^2 is expected to be approximately

$$\delta k_y^2 \lesssim k_x^2\frac{\delta}{\Delta} \tag{38}$$

This error can be significant. Equation 36 can be used as a first estimate of k_y^2, but once $\widetilde{x}_1$ is determined, equation 34 should be used.

We now wish to find $\widetilde{x}_1$. To do this, we use B_x. Allowing for a probe roll angle θ_r, the expression for B_x at the field peaks becomes

$$B_x = \phi_0 k_x k_x \widetilde{x} k_y \widetilde{y} + \theta_r B_y \tag{39}$$

where θ_r is the probe roll angle. The two probes with vertical offset measure

$$(B_x)_1 = \phi_0 k_x^2 \widetilde{x}_1 k_y \widetilde{y}_1 + (\theta_r)_1 \phi_0 k_y \left[1 + \frac{1}{2}(k_x\widetilde{x}_1)^2 + \frac{1}{2}(k_y\widetilde{y}_1)^2\right] \tag{40}$$

$$(B_x)_2 = \phi_0 k_x^2 \widetilde{x}_2 k_y \widetilde{y}_2 + (\theta_r)_2 \phi_0 k_y \left[1 + \frac{1}{2}(k_x\widetilde{x}_2)^2 + \frac{1}{2}(k_y\widetilde{y}_2)^2\right] \tag{41}$$

Letting $\widetilde{x}_2 = \widetilde{x}_1 + \delta$ and $\widetilde{y}_2 = \widetilde{y}_1 + \Delta$, we have

$$\begin{aligned}(B_x)_2 - (B_x)_1 &= \phi_0 k_x^2(\widetilde{x}_1 + \delta)k_y(\widetilde{y}_1 + \Delta) - \phi_0 k_x^2 \widetilde{x}_1 k_y \widetilde{y}_1 \\ &\quad + [(\theta_r)_2 - (\theta_r)_1]\phi_0 k_y + \theta_r \phi_0 k_y \left[k_x^2\left(\widetilde{x}_1\delta + \frac{1}{2}\delta^2\right) + k_y^2\left(\widetilde{y}_1\Delta + \frac{1}{2}\Delta^2\right)\right]\end{aligned} \tag{42}$$

By subtracting $(B_x)_1$ from $(B_x)_2$, we have taken out the part of the $\theta_r B_y$ mixing term due to overall probe roll. Relative misalignments of the probe elements have already been taken out in the initial conversion from Hall element signals to fields. Small errors in the angle calibration of the Hall elements remain, however. Let $[(\theta_r)_2 - (\theta_r)_1]$ for the B_x elements after angle corrections have been performed be denoted by ϵ_{x21}. Since $\phi_0 k_y = (B_y)_1$ without changing the order of the expression, we set $[(\theta_r)_2 - (\theta_r)_1]\,\phi_0 k_y = \epsilon_{x21}\,(B_y)_1$. The last term is second order inside the brackets. θ_r is the overall probe roll angle. Differences in the roll angles of the Hall elements have been corrected and errors in the corrections make negligible contributions in this term. We still must require that the last term involving overall probe roll θ_r be much smaller than the difference between the first two terms. Since the quantities in the bracket have similar magnitudes to those in the first two terms, we require $\theta_r \ll 1$. This is the case with proper alignment. We have

$$(B_x)_2 - (B_x)_1 - \epsilon_{x21}\,(B_y)_1 = \phi_0 k_x^2\,(\widetilde{x}_1 + \delta)\,k_y\,(\widetilde{y}_1 + \Delta) - \phi_0 k_x^2 \widetilde{x}_1 k_y \widetilde{y}_1 \tag{43}$$

Setting $\phi_0 k_y = (B_y)_1$, which does not change the order, we have

$$\frac{(B_x)_2 - (B_x)_1 - \epsilon_{x21}\,(B_y)_1}{(B_y)_1} = k_x^2\,(\widetilde{x}_1\Delta + \widetilde{y}_1\delta + \delta\Delta) \tag{44}$$

In order for the error term to be negligible, we require

$$\epsilon_{x21} \ll k_x^2\,(\widetilde{x}_1\Delta + \widetilde{y}_1\delta + \delta\Delta) \tag{45}$$

The right hand side can be as small as, or on the order of 10^{-4}. We thus require that the relative misalignment of the two B_x Hall probes be known to 0.1 mrad. This will be very difficult, but perhaps can be done by aligning one element to a strong field and measuring the signal on the other element, which will give the angle between the two probes. If it seems possible to measure angles accurately enough, we can solve for $\widetilde{x}_1$ to find

$$\boxed{\widetilde{x}_1 = \frac{(B_x)_2 - (B_x)_1}{(B_y)_1}\frac{1}{k_x^2\Delta} - \widetilde{y}_1\frac{\delta}{\Delta} - \delta} \tag{46}$$

The error on $\widetilde{x}_1$ from ϵ_{x21} is

$$\delta\widetilde{x}_1 = \frac{\epsilon_{x21}}{k_x^2\Delta} \tag{47}$$

This error can be large. For example, if $\epsilon_{x21} = 0.1$ mrad, $k_x = 100$ 1/m, and $\Delta = 10^{-4}$ m, then $\delta\widetilde{x}_1 = 100$ microns. The actual case in the undulator will need to be evaluated. We know all the terms in the expression for $\widetilde{x}_1$, acknowledging that k_x^2 was not measured, but rather came from the constraint $k_x^2 = k_u^2 - k_y^2$. If we can determine $\widetilde{x}_1$, we can make the more accurate determination of k_y^2 in equation 34, and in this way iterate to find more accurate values of k_x^2 and $\widetilde{x}_1$.

4.1.2 Probes Offset In The x-Direction

We now rotate the probe body so the probes are offset along the x-axis. When we do this, the probe positions relative to the magnetic center change since the Hall elements are not precisely on the axis of rotation. With the offset direction horizontal, let $\overline{x} = x - x_c$, where x is the x-position the field is evaluated at (the probe location) and x_c is the x-position of the magnetic center. Similarly, $\overline{y} = y - y_c$, where y is the y-position the field is evaluated at and y_c is the y-position of the magnetic center.

In the linear polarization vertical field mode, the fields at the probe locations are

$$B_x = \phi_0 k_x \sinh(k_x\overline{x})\sinh(k_y\overline{y})\cos(k_u z) \tag{48}$$
$$B_y = \phi_0 k_y \cosh(k_x\overline{x})\cosh(k_y\overline{y})\cos(k_u z) \tag{49}$$
$$B_z = -\phi_0 k_u \cosh(k_x\overline{x})\sinh(k_y\overline{y})\sin(k_u z) \tag{50}$$

Expanding to first order in $k_x\overline{x}$ and $k_y\overline{y}$, the fields become

$$B_x = 0 \tag{51}$$
$$B_y = \phi_0 k_y \cos(k_u z) \tag{52}$$
$$B_z = -\phi_0 k_u k_y \overline{y} \sin(k_u z) \tag{53}$$

Expanding to second order in $k_x\overline{x}$ and $k_y\overline{y}$, the fields become

$$B_x = \phi_0 k_x k_x \overline{x} k_y \overline{y} \cos(k_u z) \tag{54}$$
$$B_y = \phi_0 k_y \left[1 + \frac{1}{2}(k_x\overline{x})^2 + \frac{1}{2}(k_y\overline{y})^2\right] \cos(k_u z) \tag{55}$$
$$B_z = -\phi_0 k_u k_y \overline{y} \sin(k_u z) \tag{56}$$

These equations have the same form as in the previous section, but we have carefully noted that the probe positions relative to the magnetic center are different. We repeat calculations similar to those given in the previous section.

Using B_z and the linear expansions and looking at the field peaks, we obtain

$$\frac{B_z}{B_y} = -k_u \overline{y} \tag{57}$$

Applying this to probe 1, we find

$$\boxed{\overline{y}_1 = y_1 - y_c = -\frac{1}{k_u}\left(\frac{B_z}{B_y}\right)_1} \tag{58}$$

With the probes offset in the x-direction, probe 2 is at $\overline{y}_2 = \overline{y}_1 - \delta$. So

$$\overline{y}_2 = \overline{y}_1 - \delta = -\frac{1}{k_u}\left(\frac{B_z}{B_y}\right)_2 \tag{59}$$

This lets us solve for δ.

$$-\delta = -\frac{1}{k_u}\left(\frac{B_z}{B_y}\right)_2 + \frac{1}{k_u}\left(\frac{B_z}{B_y}\right)_1 \tag{60}$$

or, since $(B_y)_2 \simeq (B_y)_1$,

$$\boxed{\delta = \frac{1}{k_u}\frac{(B_z)_2 - (B_z)_1}{(B_y)_1}} \tag{61}$$

We now wish to find k_x^2. We use B_x and allow for a probe roll angle θ_r. The expression for B_x at the field peaks is

$$B_x = \phi_0 k_x k_x \overline{x} k_y \overline{y} + \theta_r B_y \tag{62}$$

The two probes with horizontal offset measure

$$(B_x)_1 = \phi_0 k_x^2 \overline{x}_1 k_y \overline{y}_1 + (\theta_r)_1 \phi_0 k_y \left[1 + \frac{1}{2}(k_x\overline{x}_1)^2 + \frac{1}{2}(k_y\overline{y}_1)^2\right] \tag{63}$$
$$(B_x)_2 = \phi_0 k_x^2 \overline{x}_2 k_y \overline{y}_2 + (\theta_r)_2 \phi_0 k_y \left[1 + \frac{1}{2}(k_x\overline{x}_2)^2 + \frac{1}{2}(k_y\overline{y}_2)^2\right] \tag{64}$$

With the probes rotated so the main offset is in the x-direction, we have $\overline{y}_2 = \overline{y}_1 - \delta$ and $\overline{x}_2 = \overline{x}_1 + \Delta$. With these probe positions we have

$$\begin{aligned}(B_x)_2 - (B_x)_1 = {} & \phi_0 k_x^2 (\overline{x}_1 + \Delta) k_y (\overline{y}_1 - \delta) - \phi_0 k_x^2 \overline{x}_1 k_y \overline{y}_1 \\ & + [(\theta_r)_2 - (\theta_r)_1] \phi_0 k_y + \theta_r \phi_0 k_y \left[\frac{1}{2}k_x^2\left(2\overline{x}_1\Delta + \Delta^2\right) + \frac{1}{2}k_y^2\left(-2\overline{y}_1\delta + \delta^2\right)\right]\end{aligned} \tag{65}$$

We have eliminated the main effect of probe roll by taking the difference of the measurements. Errors in the angle calibration of the Hall elements remain. Let $[(\theta_r)_2 - (\theta_r)_1]$ for the B_x elements after the angle calibration has been applied be denoted by ϵ_{x21}. Since $\phi_0 k_y = (B_y)_1$ without changing the order of the expression, we set $[(\theta_r)_2 - (\theta_r)_1]\,\phi_0 k_y = \epsilon_{x21}(B_y)_1$. The last term is second order inside the brackets. θ_r is the overall probe roll angle. Differences in the roll angles of the Hall elements have been corrected and errors in the corrections make negligible contributions in this term. In order to eliminate the last term from consideration, we require $\theta_r \ll 1$. Setting $\phi_0 k_y = (B_y)_1$ in the first two terms on the right, which does not change the order, we find

$$\begin{aligned} \frac{(B_x)_2 - (B_x)_1 - \epsilon_{x21}(B_y)_1}{(B_y)_1} &= k_x^2\left[(\overline{x}_1 + \Delta)(\overline{y}_1 - \delta) - \overline{x}_1\overline{y}_1\right] \\ &= k_x^2\left(-\overline{x}_1\delta + \overline{y}_1\Delta - \delta\Delta\right) \end{aligned} \tag{66}$$

As in the previous section, we require the angle calibration of Hall elements to be accurate to 0.1 mrad for the unknown ϵ_{x21} term to be negligible. If this is the case,

$$\boxed{k_x^2 = \frac{(B_x)_2 - (B_x)_1}{(B_y)_1}\frac{1}{(\overline{y}_1\Delta - \overline{x}_1\delta - \delta\Delta)}} \tag{67}$$

Since the probe was constructed with $\delta \ll \Delta$, we can start an iterative evaluation of k_x^2 by taking

$$k_x^2 = \frac{(B_x)_2 - (B_x)_1}{(B_y)_1}\frac{1}{\overline{y}_1\Delta} \tag{68}$$

We know $\overline{y}_1$ from equation 58 and we know Δ. We can also calculate k_y^2 since

$$k_y^2 = k_u^2 - k_x^2 \tag{69}$$

We now wish to determine $\overline{x}_1$. To do this, we use the second order expansion of B_y. With the probes aligned so the offset is in the x-direction, we have

$$\overline{x}_2 = \overline{x}_1 + \Delta \tag{70}$$

$$\overline{y}_2 = \overline{y}_1 - \delta \tag{71}$$

Probe 2 measures peak fields with values

$$(B_y)_2 = \phi_0 k_y\left[1 + \frac{1}{2}(k_x\overline{x}_2)^2 + \frac{1}{2}(k_y\overline{y}_2)^2\right] \tag{72}$$

$$= \phi_0 k_y\left[1 + \frac{1}{2}k_x^2\left(\overline{x}_1^2 + 2\overline{x}_1\Delta + \Delta^2\right) + \frac{1}{2}k_y^2\left(\overline{y}_1^2 - 2\overline{y}_1\delta + \delta^2\right)\right] \tag{73}$$

$$= (B_y)_1 + \phi_0 k_y\left[\frac{1}{2}k_x^2\left(2\overline{x}_1\Delta + \Delta^2\right) + \frac{1}{2}k_y^2\left(-2\overline{y}_1\delta + \delta^2\right)\right] \tag{74}$$

Working to second order, we find

$$\frac{(B_y)_2 - (B_y)_1}{(B_y)_1} = k_x^2\left(\overline{x}_1\Delta + \frac{1}{2}\Delta^2\right) + k_y^2\left(-\overline{y}_1\delta + \frac{1}{2}\delta^2\right) \tag{75}$$

We solve this for $\overline{x}_1$.

$$\boxed{\overline{x}_1 = \frac{(B_y)_2 - (B_y)_1}{(B_y)_1}\frac{1}{k_x^2\Delta} - \frac{1}{2}\Delta - \frac{k_y^2}{k_x^2}\left(-\overline{y}_1 + \frac{1}{2}\delta\right)\frac{\delta}{\Delta}} \tag{76}$$

We know or have first estimates for all the quantities on the right hand side of this expression. We can go back to equation 67 and insert this value of $\overline{x}_1$ to obtain a more accurate calculation for k_x^2, k_y^2, and $\overline{x}_1$, and continue iterating to find a solution.

4.1.3 Linear Polarization Vertical Field Summary

With the probes offset vertically, we determine $\widetilde{y}_1$, Δ, k_y^2, and calculate k_x^2 from the constraint. An accurate value for $\widetilde{x}_1$ is required in the calculation of k_y. If the Hall element angles are calibrated at the 0.1 mrad level, we determine $\widetilde{x}_1$.

With the probes offset horizontally, we determine $\overline{y}_1$, δ, and $\overline{x}_1$, however, an accurate value of k_x^2 is required in this calculation since $\overline{x}_1 \sim 1/k_x^2$. If the Hall element angles are calibrated at the 0.1 mrad level, we determine k_x^2.

4.2 Linear Polarization Horizontal Field

In the following discussion, we use the symbols k_x and k_y, but they are for the horizontal field mode, and they are different than k_x and k_y in the vertical field mode. On the other hand, we expect by symmetry that k_x in the horizontal field mode is the same as k_y in the vertical field mode. Similarly, k_y in the horizontal field mode should be the same as k_x in the vertical field mode.

Since we assume that the magnetic center does not change with polarization mode, we expect $\widetilde{x}_1$, $\widetilde{y}_1$, $\overline{x}_1$, and $\overline{y}_1$ to be the same in the horizontal field and in the vertical field modes. This provides a valuable check.

Also, since Δ and δ are inherent to the probe, we expect them to be the same in all modes they are calculated in.

4.2.1 Probes Offset In The x-Direction

Consider the linear polarization horizontal field mode of the undulator. Suppose the probes are offset along the x-axis. Let $\overline{x} = x - x_c$ and $\overline{y} = y - y_c$, as defined above.

The fundamental terms in the field expansion are

$$B_x = \phi_0 k_x \cosh(k_x \overline{x}) \cosh(k_y \overline{y}) \cos(k_u z) \tag{77}$$
$$B_y = \phi_0 k_y \sinh(k_x \overline{x}) \sinh(k_y \overline{y}) \cos(k_u z) \tag{78}$$
$$B_z = -\phi_0 k_u \sinh(k_x \overline{x}) \cosh(k_y \overline{y}) \sin(k_u z) \tag{79}$$

Expanding to first order in $k_x\overline{x}$ and $k_y\overline{y}$, the fields become

$$B_x = \phi_0 k_x \cos(k_u z) \tag{80}$$
$$B_y = 0 \tag{81}$$
$$B_z = -\phi_0 k_u k_x \overline{x} \sin(k_u z) \tag{82}$$

Expanding to second order in $k_x\overline{x}$ and $k_y\overline{y}$, the fields become

$$B_x = \phi_0 k_x \left[1 + \frac{1}{2}(k_x \overline{x})^2 + \frac{1}{2}(k_y \overline{y})^2\right] \cos(k_u z) \tag{83}$$
$$B_y = \phi_0 k_y k_x \overline{x} k_y \overline{y} \cos(k_u z) \tag{84}$$
$$B_z = -\phi_0 k_u k_x \overline{x} \sin(k_u z) \tag{85}$$

Using the linear expansions and looking at the field peaks, we obtain

$$\frac{B_z}{B_x} = -k_u \overline{x} \tag{86}$$

Applying this to probe 1, we find

$$\boxed{\overline{x}_1 = x_1 - x_c = -\tfrac{1}{k_u}\left(\tfrac{B_z}{B_x}\right)_1} \tag{87}$$

We determine the x-position of probe 1 relative to the magnetic center when the probe offset is horizontal by using the ratio of the B_z and B_x peak fields from probe 1. $k_u = 2\pi/\lambda_u$ is known. In order to get the sign correct in this expression, we note that $B_z \sim \sin(k_u z)$ and $B_x \sim \cos(k_u z)$, so we must use the peak of B_z at the z-position $\lambda_u/4$ past the peak of B_x.

If we do a similar analysis for probe 2, we find

$$\overline{x}_2 = x_2 - x_c = -\frac{1}{k_u}\left(\frac{B_z}{B_x}\right)_2 \tag{88}$$

With the probes aligned so that the probes are offset in the x-direction, probe 2 is at $x_2 = x_1 + \Delta$, where Δ is the probe offset distance. Subtracting, we find

$$\Delta = -\frac{1}{k_u}\left[\left(\frac{B_z}{B_x}\right)_2 - \left(\frac{B_z}{B_x}\right)_1\right] \tag{89}$$

or since $(B_x)_2 = (B_x)_1$ to first order,

$$\boxed{\Delta = -\frac{1}{k_u}\frac{(B_z)_2 - (B_z)_1}{(B_x)_1}} \tag{90}$$

The offset distance between the probes will be measured independently using fiducialization magnets, and the offset distance will be compared to the result calculated above using the fields.

Consider the second order expansions of the fields with the probes offset in the x-direction. Let

$$\overline{x}_2 = \overline{x}_1 + \Delta \tag{91}$$

$$\overline{y}_2 = \overline{y}_1 - \delta \tag{92}$$

where we know Δ and δ. Ideally, $\delta = 0$ so the probes have the same y-position, but in general we expect $\delta \ll \Delta$. Probe 2 measures peak fields with values

$$(B_x)_2 = \phi_0 k_x \left[1 + \frac{1}{2}(k_x \overline{x}_2)^2 + \frac{1}{2}(k_y \overline{y}_2)^2\right] \tag{93}$$

$$= \phi_0 k_x \left[1 + \frac{1}{2}k_x^2\left(\overline{x}_1^2 + 2\overline{x}_1\Delta + \Delta^2\right) + \frac{1}{2}k_y^2\left(\overline{y}_1^2 - 2\overline{y}_1\delta + \delta^2\right)\right] \tag{94}$$

$$= (B_x)_1 + \phi_0 k_x \left[\frac{1}{2}k_x^2\left(2\overline{x}_1\Delta + \Delta^2\right) + \frac{1}{2}k_y^2\left(-2\overline{y}_1\delta + \delta^2\right)\right] \tag{95}$$

Working to second order, we find

$$\frac{(B_x)_2 - (B_x)_1}{(B_x)_1} = \left[k_x^2\left(\overline{x}_1\Delta + \frac{1}{2}\Delta^2\right) + k_y^2\left(-\overline{y}_1\delta + \frac{1}{2}\delta^2\right)\right] \tag{96}$$

As a first approximation, we assume the second term on the right is small, so that

$$k_y^2\left(-\overline{y}_1\delta + \frac{1}{2}\delta^2\right) \ll k_x^2\left(\overline{x}_1\Delta + \frac{1}{2}\Delta^2\right) \tag{97}$$

When this is the case,

$$\frac{(B_x)_2 - (B_x)_1}{(B_x)_1} = k_x^2\left(\overline{x}_1\Delta + \frac{1}{2}\Delta^2\right) \tag{98}$$

and we can solve for k_x^2,

$$k_x^2 = \frac{(B_x)_2 - (B_x)_1}{(B_x)_1}\frac{1}{\left(\overline{x}_1\Delta + \frac{1}{2}\Delta^2\right)} \tag{99}$$

We know $\overline{x}_1$ from equation 87 and we know Δ, so all terms on the right hand side are known. When we know $\overline{y}_1$ and k_y^2, we can make a more accurate calculation to obtain

$$\boxed{k_x^2 = \frac{(B_x)_2-(B_x)_1}{(B_x)_1}\frac{1}{\left(\overline{x}_1\Delta+\frac{1}{2}\Delta^2\right)} - k_y^2\frac{\left(-\overline{y}_1\delta+\frac{1}{2}\delta^2\right)}{\left(\overline{x}_1\Delta+\frac{1}{2}\Delta^2\right)}} \tag{100}$$

The constraint on the k's, $k_x^2 + k_y^2 = k_u^2$, allows us to determine k_y^2 once we know k_x^2.

$$k_y^2 = k_u^2 - k_x^2 \tag{101}$$

In order to find $\overline{y}_1$, we use B_y. Allowing for a probe roll angle θ_r, the expression for B_y at the field peaks becomes

$$B_y = \phi_0 k_y k_x \overline{x} k_y \overline{y} + \theta_r B_x \tag{102}$$

The two probes with horizontal offset measure

$$(B_y)_1 = \phi_0 k_x \overline{x}_1 k_y^2 \overline{y}_1 + (\theta_r)_1 \phi_0 k_x \left[1 + \frac{1}{2}(k_x\overline{x}_1)^2 + \frac{1}{2}(k_y\overline{y}_1)^2\right] \tag{103}$$

$$(B_y)_2 = \phi_0 k_x \overline{x}_2 k_y^2 \overline{y}_2 + (\theta_r)_2 \phi_0 k_x \left[1 + \frac{1}{2}(k_x\overline{x}_2)^2 + \frac{1}{2}(k_y\overline{y}_2)^2\right] \tag{104}$$

With the probes oriented horizontally, $\overline{x}_2 = \overline{x}_1 + \Delta$ and $\overline{y}_2 = \overline{y}_1 - \delta$. The difference in the B_y measurements is

$$\begin{aligned}(B_y)_2 - (B_y)_1 &= \phi_0 k_x (\overline{x}_1 + \Delta) k_y^2 (\overline{y}_1 - \delta) - \phi_0 k_x \overline{x}_1 k_y^2 \overline{y}_1 + [(\theta_r)_2 - (\theta_r)_1] \phi_0 k_x \\ &\quad + \theta_r \phi_0 k_x \left[\frac{1}{2}k_x^2\left(2\overline{x}_1\Delta + \Delta^2\right) + \frac{1}{2}k_y^2\left(-2\overline{y}_1\delta + \delta^2\right)\right]\end{aligned} \tag{105}$$

The large contribution from the overall probe roll angle has cancelled in the difference. Hall element angles have been measured in a calibration and corrected. Let $[(\theta_r)_2 - (\theta_r)_1]$ for the B_y Hall elements be the error in the Hall element angle calibration denoted by ϵ_{y21}. Since $\phi_0 k_x = (B_x)_1$ without changing the order of the expression, we set $[(\theta_r)_2 - (\theta_r)_1] \phi_0 k_x = \epsilon_{y21} (B_x)_1$. The last term is second order inside the brackets. θ_r is the overall probe roll angle. Differences in the roll angles of the Hall elements have been corrected and errors in the corrections make negligible contributions in this term. We require the overall probe roll $\theta_r \ll 1$ so the last terms become insignificant. In this case

$$(B_y)_2 - (B_y)_1 - \epsilon_{y21} (B_x)_1 = \phi_0 k_x k_y^2 (-\overline{x}_1\delta + \overline{y}_1\Delta - \delta\Delta) \tag{106}$$

Setting $\phi_0 k_x = (B_x)_1$ without changing the order, we find

$$\frac{(B_y)_2 - (B_y)_1 - \epsilon_{y21} (B_x)_1}{(B_x)_1} = k_y^2 (-\overline{x}_1\delta + \overline{y}_1\Delta - \delta\Delta) \tag{107}$$

As noted in the previous section, we require the Hall element angle calibration to be accurate to 0.1 mrad in order for the unknown ϵ_{y21} error term to be negligible. If this is the case,

$$\boxed{\overline{y}_1 = \frac{(B_y)_2-(B_y)_1}{(B_x)_1}\frac{1}{k_y^2\Delta} + \overline{x}_1\frac{\delta}{\Delta} + \delta} \tag{108}$$

We know all the quantities on the right hand side of this expression.

4.2.2 Probes Offset In The y-Direction

We now rotate the probe body so the probes are offset along the y-axis. The probe positions relative to the magnetic center should be the same as when the undulator was in vertical field mode. We go back to the tilde notation. With the offset direction vertical, let $\widetilde{x} = x - x_c$ and $\widetilde{y} = y - y_c$, as defined above.

In the linear polarization horizontal field mode, the fields at the probe locations are

$$\begin{aligned} B_x &= \phi_0 k_x \cosh(k_x\widetilde{x})\cosh(k_y\widetilde{y})\cos(k_u z) && (109)\\ B_y &= \phi_0 k_y \sinh(k_x\widetilde{x})\sinh(k_y\widetilde{y})\cos(k_u z) && (110)\\ B_z &= -\phi_0 k_u \sinh(k_x\widetilde{x})\cosh(k_y\widetilde{y})\sin(k_u z) && (111)\end{aligned}$$

Expanding to first order in $k_x\widetilde{x}$ and $k_y\widetilde{y}$, the fields become

$$\begin{aligned} B_x &= \phi_0 k_x \cos(k_u z) && (112)\\ B_y &= 0 && (113)\\ B_z &= -\phi_0 k_u k_x \widetilde{x}\sin(k_u z) && (114)\end{aligned}$$

Expanding to second order in $k_x\widetilde{x}$ and $k_y\widetilde{y}$, the fields become

$$\begin{aligned} B_x &= \phi_0 k_x \left[1 + \frac{1}{2}(k_x\widetilde{x})^2 + \frac{1}{2}(k_y\widetilde{y})^2\right]\cos(k_u z) && (115)\\ B_y &= \phi_0 k_y k_x \widetilde{x} k_y \widetilde{y}\cos(k_u z) && (116)\\ B_z &= -\phi_0 k_u k_x \widetilde{x}\sin(k_u z) && (117)\end{aligned}$$

These equations have the same form as in the previous section, but we have noted by the tilde that the probe positions relative to the magnetic center are different. We repeat calculations similar to those given in the previous section.

Using B_z and B_x with the linear expansion, and looking at the field peaks, we obtain

$$\frac{B_z}{B_x} = -k_u\widetilde{x} \qquad (118)$$

Applying this to probe 1, we find

$$\boxed{\widetilde{x}_1 = x_1 - x_c = -\frac{1}{k_u}\left(\frac{B_z}{B_x}\right)_1} \qquad (119)$$

We can also apply this to probe 2 where $\widetilde{x}_2 = \widetilde{x}_1 + \delta$.

$$\widetilde{x}_2 = \widetilde{x}_1 + \delta = -\frac{1}{k_u}\left(\frac{B_z}{B_x}\right)_2 \qquad (120)$$

Subtracting, we find to second order,

$$\boxed{\delta = -\frac{1}{k_u}\frac{(B_z)_2 - (B_z)_1}{(B_x)_1}} \qquad (121)$$

We wish to determine k_y^2. Using B_y and allowing for a probe roll angle θ_r, the expression for B_y at the field peaks becomes

$$B_y = \phi_0 k_y k_x \widetilde{x} k_y \widetilde{y} + \theta_r B_x \qquad (122)$$

The two probes with vertical offset measure

$$(B_y)_1 = \phi_0 k_x \widetilde{x}_1 k_y^2 \widetilde{y}_1 + (\theta_r)_1 \phi_0 k_x \left[1 + \frac{1}{2}(k_x \widetilde{x}_1)^2 + \frac{1}{2}(k_y \widetilde{y}_1)^2\right] \tag{123}$$

$$(B_y)_2 = \phi_0 k_x \widetilde{x}_2 k_y^2 \widetilde{y}_2 + (\theta_r)_2 \phi_0 k_x \left[1 + \frac{1}{2}(k_x \widetilde{x}_2)^2 + \frac{1}{2}(k_y \widetilde{y}_2)^2\right] \tag{124}$$

Probe 2 is at $\widetilde{x}_2 = \widetilde{x}_1 + \delta$ and $\widetilde{y}_2 = \widetilde{y}_1 + \Delta$. Inserting these values and subtracting, we find

$$\begin{aligned}(B_y)_2 - (B_y)_1 &= \phi_0 k_x (\widetilde{x}_1 + \delta) k_y^2 (\widetilde{y}_1 + \Delta) - \phi_0 k_x \widetilde{x}_1 k_y^2 \widetilde{y}_1 \\ &\quad + [(\theta_r)_2 - (\theta_r)_1] \phi_0 k_x + \theta_r \phi_0 k_x \left[\frac{1}{2} k_x^2 \left(2\widetilde{x}_1 \delta + \delta^2\right) + \frac{1}{2} k_y^2 \left(2\widetilde{y}_1 \Delta + \Delta^2\right)\right]\end{aligned} \tag{125}$$

Subtracting the fields from the two probes has eliminated the main coupling from probe roll of B_x into the difference. Angle errors of the Hall elements have been calibrated and corrected. Errors in the calibration remain, however. Let $[(\theta_r)_2 - (\theta_r)_1]$ for the B_y element's angle errors be denoted by ϵ_{y21}. Since $\phi_0 k_x = (B_x)_1$ without changing the order of the expression, we set $[(\theta_r)_2 - (\theta_r)_1] \phi_0 k_x = \epsilon_{y21} (B_x)_1$. The last term is second order inside the brackets. θ_r is the overall probe roll angle. Differences in the roll angles of the Hall elements have been corrected and errors in the corrections make negligible contributions in this term. Since the last term in brackets has the same magnitude at the first two terms on the right hand side, we still require that overall probe roll $\theta_r \ll 1$. This lets us neglect the bracketed term. With these requirements, we have

$$\begin{aligned}(B_y)_2 - (B_y)_1 - \epsilon_{y21} (B_x)_1 &= \phi_0 k_x (\widetilde{x}_1 + \delta) k_y^2 (\widetilde{y}_1 + \Delta) - \phi_0 k_x \widetilde{x}_1 k_y^2 \widetilde{y}_1 \\ &= \phi_0 k_x k_y^2 (\widetilde{x}_1 \Delta + \widetilde{y}_1 \delta + \delta \Delta)\end{aligned} \tag{126}$$

We know $\widetilde{x}_1$, so we can solve for k_y^2. Using $\phi_0 k_x = (B_x)_1$ without changing the order of the expression, we have

$$\frac{(B_y)_2 - (B_y)_1 - \epsilon_{y21} (B_x)_1}{(B_x)_1} = k_y^2 (\widetilde{x}_1 \Delta + \widetilde{y}_1 \delta + \delta \Delta) \tag{127}$$

As noted in the previous section, we require the Hall element angle calibration to be accurate to 0.1 mrad in order for the unknown ϵ_{y21} error term to be negligible. If this is the case,

$$\boxed{k_y^2 = \frac{(B_y)_2 - (B_y)_1}{(B_x)_1} \frac{1}{(\widetilde{x}_1 \Delta + \widetilde{y}_1 \delta + \delta \Delta)}} \tag{128}$$

We can use this expression iteratively since $\delta \ll \Delta$. For the first iteration we set $\delta = 0$ to obtain

$$k_y^2 = \frac{(B_y)_2 - (B_y)_1}{(B_x)_1} \frac{1}{\widetilde{x}_1 \Delta} \tag{129}$$

We know $\widetilde{x}_1$ from equation 119 and we know Δ, so k_y^2 is determined for the first iteration if probe roll is acceptable.

Knowing k_y^2 gives us k_x^2 from the constraint $k_x^2 + k_y^2 = k_u^2$.

$$k_x^2 = k_u^2 - k_y^2 \tag{130}$$

We wish to determine $\widetilde{y}_1$ from the second order expansion of B_x. With the probes aligned so the offset is in the y-direction, we have

$$\widetilde{x}_2 = \widetilde{x}_1 + \delta \tag{131}$$

$$\widetilde{y}_2 = \widetilde{y}_1 + \Delta \tag{132}$$

Probe 2 measures peak fields with values

$$(B_x)_2 = \phi_0 k_x \left[1 + \frac{1}{2}(k_x \widetilde{x}_2)^2 + \frac{1}{2}(k_y \widetilde{y}_2)^2\right] \tag{133}$$

$$= \phi_0 k_x \left[1 + \frac{1}{2}k_x^2 \left(\widetilde{x}_1^2 + 2\widetilde{x}_1\delta + \delta^2\right) + \frac{1}{2}k_y^2 \left(\widetilde{y}_1^2 + 2\widetilde{y}_1\Delta + \Delta^2\right)\right] \tag{134}$$

$$= (B_x)_1 + \phi_0 k_x \left[k_x^2 \left(\widetilde{x}_1\delta + \frac{1}{2}\delta^2\right) + k_y^2 \left(\widetilde{y}_1\Delta + \frac{1}{2}\Delta^2\right)\right] \tag{135}$$

Setting $\phi_0 k_x = (B_x)_1$ without changing the order of the expression, we have

$$\frac{(B_x)_2 - (B_x)_1}{(B_x)_1} = k_x^2 \left(\widetilde{x}_1\delta + \frac{1}{2}\delta^2\right) + k_y^2 \left(\widetilde{y}_1\Delta + \frac{1}{2}\Delta^2\right) \tag{136}$$

Solving for $\widetilde{y}_1$, we have

$$\boxed{\widetilde{y}_1 = \frac{(B_x)_2 - (B_x)_1}{(B_x)_1}\frac{1}{k_y^2\Delta} - \frac{1}{2}\Delta - \frac{k_x^2}{k_y^2}\left(\widetilde{x}_1 + \frac{1}{2}\delta\right)\frac{\delta}{\Delta}} \tag{137}$$

All quantities on the right hand side are known. This value of $\widetilde{y}_1$ can be used in 128 for a more accurate estimate of k_y^2, which can be used for a more accurate estimate of k_x^2 and $\widetilde{y}_1$ in an iterative fashion.

4.2.3 Linear Polarization Horizontal Field Summary

With the probes offset horizontally, we determine $\overline{x}_1$ and Δ. If $\overline{y}_1$ is known, we determine k_x^2, and through the constraint k_y^2. If the Hall element angles are calibrated at the 0.1 mrad level, we determine $\overline{y}_1$.

With the probes offset vertically, we determine $\widetilde{x}_1$ and δ. If the Hall element angles are calibrated at the 0.1 mrad level, we determine k_y^2, and through the constraint k_x^2. If we know k_y^2, we determine $\widetilde{y}_1$.

Note that we expect k_x^2 for horizontal field mode to be the same as k_y^2 for vertical field mode. We expect the magnetic center to remain the same in the different modes, so $\widetilde{x}_1$, $\widetilde{y}_1$, $\overline{x}_1$, and $\overline{y}_1$ should agree with the vertical field case. This provides a valuable check.

4.3 Circular Right Handed Polarization

4.3.1 Probes Offset In The y-Direction

Suppose we rotate the probe body so the probes are offset along the y-axis. The probe positions relative to the magnetic center are the same as when the undulator was in vertical field mode. We go back to the tilde notation. With the offset direction vertical, let $\widetilde{x} = x - x_c$ and $\widetilde{y} = y - y_c$, as defined previously.

The fields in the circular right handed polarization mode are

$$B_x = \frac{1}{2\sqrt{2}}\phi_0 k_u \cosh\left(\frac{1}{\sqrt{2}}k_u(\widetilde{x}+\widetilde{y})\right)\cos(k_u z) - \frac{1}{2\sqrt{2}}\phi_0 k_u \cosh\left(\frac{1}{\sqrt{2}}k_u(\widetilde{x}-\widetilde{y})\right)\sin(k_u z) \quad (138)$$

$$B_y = \frac{1}{2\sqrt{2}}\phi_0 k_u \cosh\left(\frac{1}{\sqrt{2}}k_u(\widetilde{x}+\widetilde{y})\right)\cos(k_u z) + \frac{1}{2\sqrt{2}}\phi_0 k_u \cosh\left(\frac{1}{\sqrt{2}}k_u(\widetilde{x}-\widetilde{y})\right)\sin(k_u z) \quad (139)$$

$$B_z = -\frac{1}{2}\phi_0 k_u \sinh\left(\frac{1}{\sqrt{2}}k_u(\widetilde{x}+\widetilde{y})\right)\sin(k_u z) - \frac{1}{2}\phi_0 k_u \sinh\left(\frac{1}{\sqrt{2}}k_u(\widetilde{x}-\widetilde{y})\right)\cos(k_u z) \quad (140)$$

Expanding to first order, the fields become

$$B_x = \frac{1}{2\sqrt{2}}\phi_0 k_u \cos(k_u z) - \frac{1}{2\sqrt{2}}\phi_0 k_u \sin(k_u z) \quad (141)$$

$$B_y = \frac{1}{2\sqrt{2}}\phi_0 k_u \cos(k_u z) + \frac{1}{2\sqrt{2}}\phi_0 k_u \sin(k_u z) \quad (142)$$

$$B_z = -\frac{1}{2}\phi_0 k_u \frac{1}{\sqrt{2}}k_u(\widetilde{x}+\widetilde{y})\sin(k_u z) - \frac{1}{2}\phi_0 k_u \frac{1}{\sqrt{2}}k_u(\widetilde{x}-\widetilde{y})\cos(k_u z) \quad (143)$$

These expressions can be simplified by using the following identities.

$$\cos(x)+\sin(x) = \sqrt{2}\sin\left(\frac{\pi}{4}+x\right) \quad (144)$$

$$\cos(x)-\sin(x) = \sqrt{2}\sin\left(\frac{\pi}{4}-x\right) \quad (145)$$

With these identities, the fields become

$$B_x = \frac{1}{2}\phi_0 k_u \sin\left(\frac{\pi}{4}-k_u z\right) \quad (146)$$

$$B_y = \frac{1}{2}\phi_0 k_u \sin\left(\frac{\pi}{4}+k_u z\right) \quad (147)$$

$$B_z = -\frac{1}{2}\phi_0 k_u^2 \widetilde{x}\sin\left(\frac{\pi}{4}+k_u z\right) + \frac{1}{2}\phi_0 k_u^2 \widetilde{y}\sin\left(\frac{\pi}{4}-k_u z\right) \quad (148)$$

Note that $\sin\left(\frac{\pi}{4}-k_u z\right)$ has peaks at the zeros of $\sin\left(\frac{\pi}{4}+k_u z\right)$. If we look at B_z at the locations where B_y has its peak values, we find

$$[B_z]_y = -\frac{1}{2}\phi_0 k_u^2 \widetilde{x} \quad (149)$$

We set $\frac{1}{2}\phi_0 k_u = B_y$ at the peak values. In this case

$$\frac{[B_z]_y}{B_y} = -k_u \widetilde{x} \quad (150)$$

This gives an expression for $\widetilde{x}$.

$$\boxed{\widetilde{x} = -\frac{1}{k_u}\frac{[B_z]_y}{B_y}} \quad (151)$$

Errors that couple B_y into the B_z measurement will affect this result. Hall element pitch does this, but we assume the pitch will be measured in a calibration procedure and corrected. Overall probe pitch will also couple B_y into B_z

$$\delta B_z = \theta_p B_y \tag{152}$$

and it will not be corrected. For the measurement to be accurate, we require that the pitch angle be smaller than $k_u \widetilde{x}$, which can be on the order of 10^{-2} for $k_u = 100$ 1/m and $\widetilde{x} = 100$ μm. We require that the probe pitch angle be kept below 2 mrad. The error on $\widetilde{x}$ is then $.002/k_u$, or 20 μm.

Similarly, looking at the locations of the peaks of B_x gives

$$\boxed{\widetilde{y} = \frac{1}{k_u} \frac{[B_z]_x}{B_x}} \tag{153}$$

When these formulas are applied to probe 1, they give $\widetilde{x}_1$ and $\widetilde{y}_1$.

Errors that couple B_x into the B_z measurement will affect this result. Hall element yaw does this, but we assume the yaw will be measured in a calibration procedure and corrected. Overall probe yaw will also couple B_x into B_z

$$\delta B_z = \theta_y B_x \tag{154}$$

and it will not be corrected. For the measurement to be accurate, we require that the yaw angle be smaller than $k_u \widetilde{y}$, which can be on the order of 10^{-2} for $k_u = 100$ 1/m and $\widetilde{y} = 100$ μm. We require that the probe yaw angle be kept below 2 mrad. The error on $\widetilde{y}$ is then $.002/k_u$, or 20 μm.

The linear dependence of B_z on $\widetilde{x}$ and $\widetilde{y}$ allows a check on the consistency of the probe separation. With $\widetilde{x}_2 = \widetilde{x}_1 + \delta$ and $\widetilde{y}_2 = \widetilde{y}_1 + \Delta$

$$(B_z)_2 = (B_z)_1 - \frac{1}{2}\phi_0 k_u^2 \delta \sin\left(\frac{\pi}{4} + k_u z\right) + \frac{1}{2}\phi_0 k_u^2 \Delta \sin\left(\frac{\pi}{4} - k_u z\right) \tag{155}$$

Looking at the peak locations of B_x and B_y gives

$$\boxed{\frac{\left[(B_z)_2\right]_y - \left[(B_z)_1\right]_y}{(B_y)_1} = -k_u \delta} \tag{156}$$

and

$$\boxed{\frac{\left[(B_z)_2\right]_x - \left[(B_z)_1\right]_x}{(B_x)_1} = k_u \Delta} \tag{157}$$

These equations provide checks on the measurements.

We now have $(\widetilde{x}_1, \widetilde{y}_1)$, the location of probe 1 relative to the magnetic center. These are the only quantities which we need to determine since k_u is known.

4.3.2 Probes Offset In The x-Direction

With the probes rotated so the offset direction is horizontal, let $\overline{x} = x - x_c$ and $\overline{y} = y - y_c$, as defined previously. The equations for the fields are the same as above, but with the "bar" quantities replacing the "tilde" quantities. Now, however, $\overline{x}_2 = \overline{x}_1 + \Delta$ and $\overline{y}_2 = \overline{y}_1 - \delta$. Considering only the field peaks, and using the equations given above, we have

$$\boxed{\overline{x}_1 = -\frac{1}{k_u} \frac{\left[(B_z)_1\right]_y}{(B_y)_1}} \tag{158}$$

and

$$\boxed{\overline{y}_1 = \frac{1}{k_u} \frac{\left[(B_z)_1\right]_x}{(B_x)_1}}$$

Similarly

$$\boxed{\frac{\left[(B_z)_2\right]_y - \left[(B_z)_1\right]_y}{(B_y)_1} = -k_u\Delta} \tag{159}$$

and

$$\boxed{\frac{\left[(B_z)_2\right]_x - \left[(B_z)_1\right]_x}{(B_x)_1} = -k_u\delta} \tag{160}$$

As noted above, probe pitch and yaw must be kept below 2 mrad.

4.3.3 Circular Polarization Right Hand Summary

Circular polarization provides an accurate way to find the magnetic center $\widetilde{x}_1$, $\widetilde{y}_1$, $\overline{x}_1$, and $\overline{y}_1$. It also provides consistency checks by giving Δ and δ. Probe pitch and yaw must be kept below 2 mrad.

4.4 Circular Left Handed Polarization

4.4.1 Probes Offset In The y-Direction

The fields are

$$\begin{aligned} B_x &= \frac{1}{2\sqrt{2}}\phi_0 k_u \cosh\left(\frac{1}{\sqrt{2}}k_u(\widetilde{x}+\widetilde{y})\right)\cos(k_u z) \\ &\quad + \frac{1}{2\sqrt{2}}\phi_0 k_u \cosh\left(\frac{1}{\sqrt{2}}k_u(\widetilde{x}-\widetilde{y})\right)\sin(k_u z) \end{aligned} \tag{161}$$

$$\begin{aligned} B_y &= \frac{1}{2\sqrt{2}}\phi_0 k_u \cosh\left(\frac{1}{\sqrt{2}}k_u(\widetilde{x}+\widetilde{y})\right)\cos(k_u z) \\ &\quad - \frac{1}{2\sqrt{2}}\phi_0 k_u \cosh\left(\frac{1}{\sqrt{2}}k_u(\widetilde{x}-\widetilde{y})\right)\sin(k_u z) \end{aligned} \tag{162}$$

$$\begin{aligned} B_z &= -\frac{1}{2}\phi_0 k_u \sinh\left(\frac{1}{\sqrt{2}}k_u(\widetilde{x}+\widetilde{y})\right)\sin(k_u z) \\ &\quad + \frac{1}{2}\phi_0 k_u \sinh\left(\frac{1}{\sqrt{2}}k_u(\widetilde{x}-\widetilde{y})\right)\cos(k_u z) \end{aligned} \tag{163}$$

Expanding to first order, the fields become

$$B_x = \frac{1}{2\sqrt{2}}\phi_0 k_u \cos(k_u z) + \frac{1}{2\sqrt{2}}\phi_0 k_u \sin(k_u z) \tag{164}$$

$$B_y = \frac{1}{2\sqrt{2}}\phi_0 k_u \cos(k_u z) - \frac{1}{2\sqrt{2}}\phi_0 k_u \sin(k_u z) \tag{165}$$

$$B_z = -\frac{1}{2}\phi_0 k_u \frac{1}{\sqrt{2}}k_u(\widetilde{x}+\widetilde{y})\sin(k_u z) + \frac{1}{2}\phi_0 k_u \frac{1}{\sqrt{2}}k_u(\widetilde{x}-\widetilde{y})\cos(k_u z) \tag{166}$$

These expressions can be simplified by using the identities:

$$\cos(x) + \sin(x) = \sqrt{2}\sin\left(\frac{\pi}{4}+x\right) \tag{167}$$

$$\cos(x) - \sin(x) = \sqrt{2}\sin\left(\frac{\pi}{4}-x\right) \tag{168}$$

With these identities, the fields become

$$B_x = \frac{1}{2}\phi_0 k_u \sin\left(\frac{\pi}{4} + k_u z\right) \tag{169}$$

$$B_y = \frac{1}{2}\phi_0 k_u \sin\left(\frac{\pi}{4} - k_u z\right) \tag{170}$$

$$B_z = \frac{1}{2}\phi_0 k_u^2 \widetilde{x} \sin\left(\frac{\pi}{4} - k_u z\right) - \frac{1}{2}\phi_0 k_u^2 \widetilde{y} \sin\left(\frac{\pi}{4} + k_u z\right) \tag{171}$$

If we look at B_z at the locations where B_y has its peak values, we find

$$[B_z]_y = \frac{1}{2}\phi_0 k_u^2 \widetilde{x} \tag{172}$$

We set $\frac{1}{2}\phi_0 k_u = B_y$ at the peak values. In this case

$$\frac{[B_z]_y}{B_y} = k_u \widetilde{x} \tag{173}$$

This gives an expression for $\widetilde{x}$.

$$\boxed{\widetilde{x} = \frac{1}{k_u}\frac{[B_z]_y}{B_y}} \tag{174}$$

Similarly, looking at the locations of the peaks of B_x gives

$$\boxed{\widetilde{y} = -\frac{1}{k_u}\frac{[B_z]_x}{B_x}} \tag{175}$$

With $\widetilde{x}_2 = \widetilde{x}_1 + \delta$ and $\widetilde{y}_2 = \widetilde{y}_1 + \Delta$, we also have

$$\boxed{\frac{\left[(B_z)_2\right]_y - \left[(B_z)_1\right]_y}{(B_y)_1} = k_u \delta} \tag{176}$$

and

$$\boxed{\frac{\left[(B_z)_2\right]_x - \left[(B_z)_1\right]_x}{(B_x)_1} = -k_u \Delta} \tag{177}$$

As noted above, probe pitch and yaw must be kept below 2 mrad.

4.4.2 Probes Offset In The x-Direction

With the probes rotated so the offset direction is horizontal, let $\overline{x} = x - x_c$ and $\overline{y} = y - y_c$, as defined previously. The equations for the fields are the same as above, but with the "bar" quantities replacing the "tilde" quantities. Now, however, $\overline{x}_2 = \overline{x}_1 + \Delta$ and $\overline{y}_2 = \overline{y}_1 - \delta$. Considering only the field peaks, and using the equations given above, we have

$$\boxed{\overline{x}_1 = \frac{1}{k_u}\frac{\left[(B_z)_1\right]_y}{(B_y)_1}} \tag{178}$$

and

$$\boxed{\overline{y}_1 = -\frac{1}{k_u}\frac{\left[(B_z)_1\right]_x}{(B_x)_1}}$$

Similarly

$$\boxed{\frac{\left[(B_z)_2\right]_y - \left[(B_z)_1\right]_y}{(B_y)_1} = k_u \Delta} \tag{179}$$

and

$$\boxed{\frac{\left[(B_z)_2\right]_x - \left[(B_z)_1\right]_x}{(B_x)_1} = -k_u \delta} \tag{180}$$

As noted above, probe pitch and yaw must be kept below 2 mrad.

4.4.3 Circular Polarization Left Hand Summary

Circular polarization provides an accurate way to find the magnetic center $\widetilde{x}_1$, $\widetilde{y}_1$, $\overline{x}_1$, and $\overline{y}_1$. It also provides consistency checks by giving Δ and δ. Probe pitch and yaw must be kept below 2 mrad.

5 Simulation

A simulation of the Hall probe array measurements was performed. The undulator fields were generated according to the fundamental terms in the longitudinal expansion presented in this note. The signals from the Hall elements were calculated with Hall element misalignment angles, misalignment angle calibration errors, and probe roll, pitch, and yaw. The analysis program then took these signals and corrected them using the calibration of the Hall element misalignment angles. It then calculated the magnetic center position and undulator parameters according to the equations in this note. The results are shown in figure 3.

Column A in each set of simulations has no Hall element angular misalignments, no errors on the misalignment angle calibrations, and no probe roll, pitch, or yaw. The parameters returned by the program are the same as those that the generator of the fields used to create the fields. Column B in each set of simulations has Hall element misalignment errors. Each element of probe 1 had -20 mrad of misalignment in each of its sensing directions. Each element of probe 2 had $+20$ mrad of misalignment in each of its sensing directions. In addition, the overall probe had 2 mrad of roll, 2 mrad of pitch, and 2 mrad of yaw. The analysis program only corrected the Hall element misalignment angles that would come from a calibration. The overall probe angles are assumed unknown and are not corrected. Column C in each set of simulations is the same as column B, with the addition that each Hall element misalignment angle has a 0.1 mrad error in the calibration which determined the angle.

The simulation results show that in the linear polarization modes, the corrections for the Hall element angular misalignments are not accurate enough if additional angles from probe roll, pitch, and yaw are added to the Hall element angles. In this note, Hall element angle errors were considered in isolation, without additional probe angles adding to their effect. The items marked in yellow have large errors and must be used with this understanding. All other quantities are fairly robust if they are measured with the angle limits specified in this note.

6 Conclusion

A plan was presented for obtaining the probe position relative to the magnetic center in the linear and circular modes of the undulator. LCLS-TN-13-4 described how to use this information to fiducialize the undulator. Furthermore, calculations were presented to determine k_x and k_y, parameters which determine the undulator transverse field behavior in the linear modes. The measurements of these parameters must be checked for consistency with the computer models of the undulator. The magnetic center measurements must be checked by verifying the consistency of results with the two probe orientations and with measurements in the different undulator modes.

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Linear Polarization Vertical Field, Probe Offset Y

	A	B	C
x1-xc (microns)	99.7	84.5	102.7
y1-yc (microns)	200	200	200
kx (1/m)	222.2	221.3	222.6
ky (1/m)	i104	i102	i104.9
Delta (microns)	100	100	100
delta (microns)			

Linear Polarization Vertical Field, Probe Offset X

	A	B	C
x1-xc (microns)	100	99.4	82.7
y1-yc (microns)	200	200	200
kx (1/m)	222	222.2	233.3
ky (1/m)	i103.7	i104.0	i126.1
Delta (microns)			
delta (microns)	15	15	15

Linear Polarization Horizontal Field, Probe Offset Y

	A	B	C
x1-xc (microns)	100	100	100
y1-yc (microns)	199.6	241.8	319.7
kx (1/m)	i103.9	i57.3	77.1
ky (1/m)	222.1	204.6	180.6
Delta (microns)			
delta (microns)	15	15	15

Linear Polarization Horizontal Field, Probe Offset X

	A	B	C
x1-xc (microns)	100	100	100
y1-yc (microns)	199.6	200.1	176.8
kx (1/m)	i104.0	i103.4	i108.1
ky (1/m)	222.2	221.9	222.4
Delta (microns)	100	100	100
delta (microns)			

Right Circular Polarization, Probe Offset Y

	A	B	C
x1-xc (microns)	99.9	90.3	90.3
y1-yc (microns)	199.9	189.3	189.3
Delta (microns)	100	100	100
delta (microns)	15	15	14

Right Circular Polarization, Probe Offset X

	A	B	C
x1-xc (microns)	99.9	90.3	90.3
y1-yc (microns)	199.9	189.3	189.3
Delta (microns)	100	99	99
delta (microns)	15	15	15

Left Circular Polarization, Probe Offset Y

	A	B	C
x1-xc (microns)	99.9	110.3	110.3
y1-yc (microns)	199.9	210.1	210.1
Delta (microns)	100	99	100
delta (microns)	15	15	16

Left Circular Polarization, Probe Offset X

	A	B	C
x1-xc (microns)	99.9	110.3	110.3
y1-yc (microns)	199.9	210.1	210.1
Delta (microns)	100	100	101
delta (microns)	15	16	15

Figure 3: Results of simulating the undulator field measurements with the Hall probe array.