

Calculation Of Fields On The Beam Axis Of The Delta Undulator

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Abstract

It is not possible to do Hall probe measurements on the beam axis of the Delta undulator since no transverse probe motion is possible. The fields at the measurement location must be used to calculate the fields on the beam axis. This note discusses the method used to do the calculation.

1 Introduction¹

Undulators are typically characterized by measuring with a Hall probe on their magnetic center line, which becomes the beam axis. The Hall probe is on stages and it is moved to the magnetic center line. The beam axis then becomes the Hall probe path, and the undulator is fiducialized to this axis. The measurements are done on the beam axis, so all quantities calculated from the measurements apply to the beam axis. These techniques do not work for the Delta undulator. The probe position is set by a guide tube and the probe follows the guide tube in the transverse directions. It is not possible to move the Hall probe to the beam axis, rather, the fields on the beam axis must be calculated from the measured fields. This note discusses the calculations.

LCLS technical note LCLS-TN-13-4² presented a measurement plan for the Delta undulator. Technical note LCLS-TN-13-9³ further discussed the Hall probe array measurements. These notes lay out the procedure for measuring the Delta undulator and determining the beam axis. This note details the procedure for taking the measured field at the measurement location, and calculating the field on the beam axis.

The Delta undulator measurements proceed as follows. The Hall probe array follows a curved path through the undulator and measures the three field components on this path. This is illustrated in figure 1. The path of the probes is found relative to a straight line by a laser system that measures transverse position changes. The probes are found relative to fiducials at the two ends of the undulator using high gradient fiducialization magnets. The probe positions at the two ends of the undulator define a line and the probe position is calculated relative to this line using the laser measurements. We take this line to be the axis of the measurement coordinate system. In the figure, y_1 is the position of probe 1 relative to the coordinate system axis. The measurements from the array of Hall elements in the probe allow us to calculate the position of the magnetic center relative to the probe. This is denoted by $y_c - y_1$ in the figure. The subscript "c" refers to the magnetic center, and the subscript "1" refers to Hall probe 1 in the probe package. Probe 1 is the probe mounted on the axis of the probe assembly. To make the calculation of the probe 1 position

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²Z. Wolf, "A Magnetic Measurement Plan For The Delta Undulator", LCLS-TN-13-4, March, 2013.

³Z. Wolf, "Hall Probe Array Measurements Of The Delta Undulator", LCLS-TN-13-9, November, 2013.

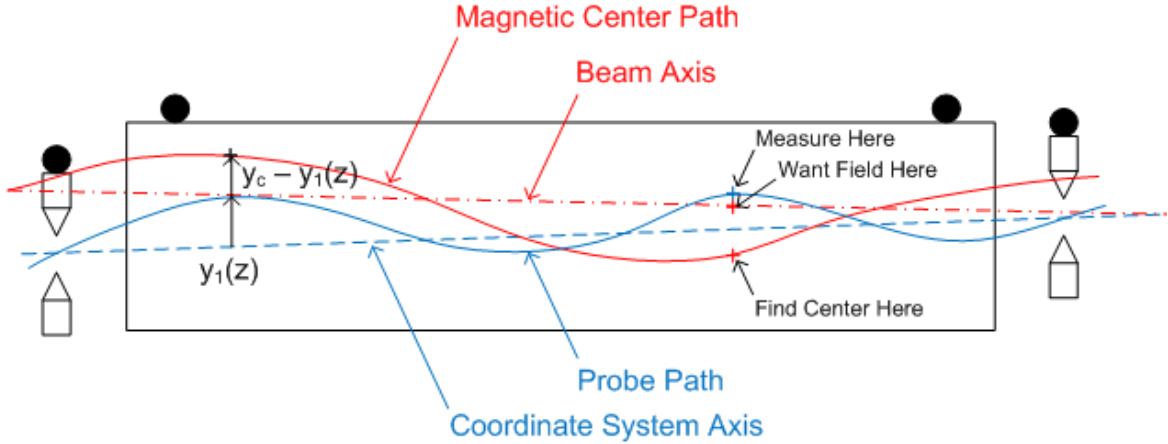


Figure 1: The field is measured on the probe path. The probe path is known relative to the coordinate system axis which is defined by the probe positions at the two ends of the undulator. The magnetic center position is calculated from the measurements. A linear fit to the magnetic centers defines the beam axis. The measured fields and the functional form of the fields allows the fields to be calculated on the beam axis.

relative to the magnetic center, the functional form of the fields must be specified. All parameters in the functional form must be calculated from the measurements. The way the functional form is determined is that Maxwell's equations are used to calculate the form of the terms in a series expansion of the field. We assume that near the magnetic center, there is a largest term at the fundamental longitudinal frequency and that it dominates the series expansion. This term is what we use for the functional form of the field. An expansion in the transverse coordinates is made assuming the transverse coordinates are close enough to the magnetic center that a second order expansion is adequate. We approximate the form of the transverse expansion as the form of the fields. Basically, we are doing a second order expansion in the transverse coordinates of the fields, and we use the Maxwell's equation solution to guide the form of the quadratic. As long as we are close enough to the magnetic center for the second order expansion to be accurate, this technique should provide an adequate parameterization of the fields. The coefficients in the quadratic expansion are parameters that must be determined from the measurements. Once we know the position of the magnetic center and the parameters in the functional form of the fields, we can calculate the fields at an arbitrary point, in particular, we can calculate the fields on the beam axis. We fit the magnetic centers with a line to determine the beam axis. We fiducialize this axis. We then calculate the fields on this line to determine the undulator characteristics.

2 Calculation Of The Fields On The Beam Axis From The Measurements

In our analysis of the fields using a second order expansion in the transverse directions, the fields had the form

$$B_i = B_{i0} f_i(x, y) g_i(z) \quad (1)$$

where $i = x, y, \text{ or } z$; f is a function which depends on the polarization mode and whose form is given explicitly below for the polarization modes discussed; and g is a function giving the longitudinal

behavior of the field. The field component measured by probe 1 is

$$B_{i1} = B_{i0}f_i(x_1 - x_c, y_1 - y_c)g_i(z) \quad (2)$$

where (x_c, y_c) are the coordinates of the magnetic center, (x_1, y_1) are the coordinates of probe 1, and $i = x, y,$ or z . The field amplitude at the magnetic center is then

$$B_{i0} = \frac{B_{i1}}{f_i(x_1 - x_c, y_1 - y_c)g_i(z)} \quad (3)$$

For the main field component, we find the field on the beam axis by substituting the beam axis coordinates to the function f_i . The field on the beam axis is given by

$$B_{ib} = B_{i0}f_i(x_b - x_c, y_b - y_c)g_i(z) \quad (4)$$

where (x_b, y_b) are the coordinates of the beam axis. By combining these equations, the fields on the beam axis are given by

$$B_{ib} = \frac{B_{i1}}{f_i(x_1 - x_c, y_1 - y_c)}f_i(x_b - x_c, y_b - y_c) \quad (5)$$

The measured field B_{i1} and all quantities in the function f_i are know, so the field on the beam axis is determined. For the main field component, f_i is of order 1, so there is no chance that the denominator will go to zero.

For the smaller field components, the functions f_i can go to zero, so the procedure used above will not work. Instead we correct the measured fields by using the Taylor series expansion in the transverse coordinates. In particular,

$$\begin{aligned} B_{ib} &= B_{i0}f_i(x_b - x_c, y_b - y_c)g_i(z) \\ &\simeq B_{i1} + \sum_{n=1}^{\infty} \frac{1}{n!} \left[(x_b - x_1) \frac{\partial}{\partial x} + (y_b - y_1) \frac{\partial}{\partial y} \right]^n B_{i1} \end{aligned} \quad (6)$$

This procedure keeps the measured field as the first term and adds corrections based on the derivatives of the transverse coordinates in the functional form of the fields. By keeping the measured field, the effect of small error fields, from magnetization direction errors for instance, which are not part of the predicted form of the field, are included.

3 Fields On The Beam Axis In The Different Undulator Modes

In this section, we detail the calculations for the fields on the beam axis. From LCLS-TN-13-9, we know the position of the probe relative to the magnetic center, $x_1 - x_c, y_1 - y_c$. We also know the parameters k_x and k_y which characterize the transverse behavior of the fields. Finally, we know the beam axis position (x_b, y_b) and the magnetic center position (x_c, y_c) in the measurement coordinate system.

In the following, we let $\tilde{x} = x - x_c$, where x is the x-position where the field is determined, and x_c is the x-position of the magnetic center. Similarly, $\tilde{y} = y - y_c$, where y is the y-position where the field is determined, and y_c is the y-position of the magnetic center. So $\tilde{x}_1 = x_1 - x_c, \tilde{y}_1 = y_1 - y_c$ for probe 1, and $\tilde{x}_b = x_b - x_c, \tilde{y}_b = y_b - y_c$ for the position of the beam axis relative to the magnetic center.

3.1 Linear Polarization Vertical Field Mode

Consider the linear polarization vertical field mode. The fields are

$$B_x = \phi_0 k_x \sinh(k_x \tilde{x}) \sinh(k_y \tilde{y}) \cos(k_u z) \quad (7)$$

$$B_y = \phi_0 k_y \cosh(k_x \tilde{x}) \cosh(k_y \tilde{y}) \cos(k_u z) \quad (8)$$

$$B_z = -\phi_0 k_u \cosh(k_x \tilde{x}) \sinh(k_y \tilde{y}) \sin(k_u z) \quad (9)$$

Expanding to second order in $k_x \tilde{x}$ and $k_y \tilde{y}$, the fields become

$$B_x = \phi_0 k_x k_x \tilde{x} k_y \tilde{y} \cos(k_u z) \quad (10)$$

$$B_y = \phi_0 k_y \left[1 + \frac{1}{2} (k_x \tilde{x})^2 + \frac{1}{2} (k_y \tilde{y})^2 \right] \cos(k_u z) \quad (11)$$

$$B_z = -\phi_0 k_u k_y \tilde{y} \sin(k_u z) \quad (12)$$

The fields measured by probe 1 are

$$B_{x1} = \phi_0 k_x k_x \tilde{x}_1 k_y \tilde{y}_1 \cos(k_u z) \quad (13)$$

$$B_{y1} = \phi_0 k_y \left[1 + \frac{1}{2} (k_x \tilde{x}_1)^2 + \frac{1}{2} (k_y \tilde{y}_1)^2 \right] \cos(k_u z) \quad (14)$$

$$B_{z1} = -\phi_0 k_u k_y \tilde{y}_1 \sin(k_u z) \quad (15)$$

The fields on the beam axis are

$$B_{xb} = \phi_0 k_x k_x \tilde{x}_b k_y \tilde{y}_b \cos(k_u z) \quad (16)$$

$$B_{yb} = \phi_0 k_y \left[1 + \frac{1}{2} (k_x \tilde{x}_b)^2 + \frac{1}{2} (k_y \tilde{y}_b)^2 \right] \cos(k_u z) \quad (17)$$

$$B_{zb} = -\phi_0 k_u k_y \tilde{y}_b \sin(k_u z) \quad (18)$$

Consider first the main field component B_y . We find B_{yb} given B_{y1} as

$$B_{yb} = B_{y1} \left[1 + \frac{1}{2} (k_x \tilde{x}_b)^2 + \frac{1}{2} (k_y \tilde{y}_b)^2 \right] / \left[1 + \frac{1}{2} (k_x \tilde{x}_1)^2 + \frac{1}{2} (k_y \tilde{y}_1)^2 \right] \quad (19)$$

This can be approximated as

$$B_{yb} = B_{y1} \left[1 + \frac{1}{2} (k_x \tilde{x}_b)^2 + \frac{1}{2} (k_y \tilde{y}_b)^2 - \frac{1}{2} (k_x \tilde{x}_1)^2 - \frac{1}{2} (k_y \tilde{y}_1)^2 \right] \quad (20)$$

B_y on the beam axis is given by B_y as measured by probe 1, plus corrections for the difference between the beam axis position and the position of probe 1.

Now consider B_x . Its derivative with respect to x is

$$\begin{aligned} \frac{\partial}{\partial x} B_x &= \phi_0 k_x k_x k_y \tilde{y} \cos(k_u z) \\ &= B_y k_x^2 \tilde{y} \end{aligned} \quad (21)$$

where we have used $B_y \simeq \phi_0 k_y \cos(k_u z)$. This approximation does not change the equation to the order that we are working at. Similarly,

$$\begin{aligned} \frac{\partial}{\partial y} B_x &= \phi_0 k_x k_x \tilde{x} k_y \cos(k_u z) \\ &= B_y k_x^2 \tilde{x} \end{aligned} \quad (22)$$

The mixed second derivative is given by

$$\begin{aligned}\frac{\partial}{\partial x} \frac{\partial}{\partial y} B_x &= \phi_0 k_x k_x k_y \cos(k_u z) \\ &= B_y k_x^2\end{aligned}\quad (23)$$

Using the Taylor series expansion, we find B_x at the beam axis as

$$\begin{aligned}B_{xb} &\simeq B_{x1} + (x_b - x_1) \frac{\partial}{\partial x} B_x|_1 + (y_b - y_1) \frac{\partial}{\partial y} B_x|_1 \\ &\quad + (x_b - x_1)(y_b - y_1) \frac{\partial}{\partial x} \frac{\partial}{\partial y} B_x|_1\end{aligned}\quad (24)$$

All other terms in the expansion are zero. Inserting values for the derivatives, we find

$$B_{xb} \simeq B_{x1} + B_{y1} k_x^2 [\tilde{y}_1 (x_b - x_1) + \tilde{x}_1 (y_b - y_1) + (x_b - x_1)(y_b - y_1)]\quad (25)$$

Using the fact that $(x_b - x_1) = (\tilde{x}_b - \tilde{x}_1)$ with a similar equation for y , this becomes

$$B_{xb} \simeq B_{x1} + B_{y1} k_x^2 [\tilde{y}_1 (\tilde{x}_b - \tilde{x}_1) + \tilde{x}_1 (\tilde{y}_b - \tilde{y}_1) + (\tilde{x}_b - \tilde{x}_1)(\tilde{y}_b - \tilde{y}_1)]\quad (26)$$

This expression gives B_x on the beam axis in terms of the measured fields at the probe position and in terms of quantities we know from the Hall probe array measurement analysis.

Now consider B_z . Its derivative with respect to x is zero, and with respect to y is

$$\frac{\partial}{\partial y} B_z = -\phi_0 k_u k_y \sin(k_u z)\quad (27)$$

$$= \frac{\partial}{\partial z} B_y\quad (28)$$

The Taylor series expansion of the field gives

$$B_{zb} = B_{z1} + \frac{\partial}{\partial z} B_y|_1 (\tilde{y}_b - \tilde{y}_1)\quad (29)$$

where we used $y_b - y_1 = \tilde{y}_b - \tilde{y}_1$ and we substituted $\frac{\partial}{\partial y} B_z = \frac{\partial}{\partial z} B_y$ which is seen from the form of the fields, and is also known since $\nabla \times \vec{B} = 0$. We know B_{z1} from the measurements, and we can calculate $\frac{\partial}{\partial z} B_y|_1$ since we know B_{y1} as a function of z . Both \tilde{y}_b and \tilde{y}_1 are known from the measurements with the Hall probe array.

In summary, the fields on the beam axis are given by

$$B_{xb} = B_{x1} + B_{y1} k_x^2 [\tilde{y}_1 (\tilde{x}_b - \tilde{x}_1) + \tilde{x}_1 (\tilde{y}_b - \tilde{y}_1) + (\tilde{x}_b - \tilde{x}_1)(\tilde{y}_b - \tilde{y}_1)]\quad (30)$$

$$B_{yb} = B_{y1} \left[1 + \frac{1}{2} (k_x \tilde{x}_b)^2 + \frac{1}{2} (k_y \tilde{y}_b)^2 - \frac{1}{2} (k_x \tilde{x}_1)^2 - \frac{1}{2} (k_y \tilde{y}_1)^2 \right]\quad (31)$$

$$B_{zb} = B_{z1} + \frac{\partial}{\partial z} B_y|_1 (\tilde{y}_b - \tilde{y}_1)\quad (32)$$

3.2 Linear Polarization Horizontal Field Mode

Consider the linear polarization horizontal field mode of the undulator. The fundamental terms in the field expansion are

$$B_x = \phi_0 k_x \cosh(k_x \tilde{x}) \cosh(k_y \tilde{y}) \cos(k_u z)\quad (33)$$

$$B_y = \phi_0 k_y \sinh(k_x \tilde{x}) \sinh(k_y \tilde{y}) \cos(k_u z)\quad (34)$$

$$B_z = -\phi_0 k_u \sinh(k_x \tilde{x}) \cosh(k_y \tilde{y}) \sin(k_u z)\quad (35)$$

Expanding to second order in $k_x \tilde{x}$ and $k_y \tilde{y}$, the fields become

$$B_x = \phi_0 k_x \left[1 + \frac{1}{2} (k_x \tilde{x})^2 + \frac{1}{2} (k_y \tilde{y})^2 \right] \cos(k_u z) \quad (36)$$

$$B_y = \phi_0 k_y k_x \tilde{x} k_y \tilde{y} \cos(k_u z) \quad (37)$$

$$B_z = -\phi_0 k_u k_x \tilde{x} \sin(k_u z) \quad (38)$$

Using the same techniques as for the vertical field mode, the fields on the beam axis are given by

$$B_{xb} = B_{x1} \left[1 + \frac{1}{2} (k_x \tilde{x}_b)^2 + \frac{1}{2} (k_y \tilde{y}_b)^2 - \frac{1}{2} (k_x \tilde{x}_1)^2 - \frac{1}{2} (k_y \tilde{y}_1)^2 \right] \quad (39)$$

$$B_{yb} \simeq B_{y1} + B_{x1} k_y^2 [\tilde{y}_1 (\tilde{x}_b - \tilde{x}_1) + \tilde{x}_1 (\tilde{y}_b - \tilde{y}_1) + (\tilde{x}_b - \tilde{x}_1) (\tilde{y}_b - \tilde{y}_1)] \quad (40)$$

$$B_{zb} = B_{z1} + \frac{\partial}{\partial z} B_{x1} (\tilde{x}_b - \tilde{x}_1) \quad (41)$$

3.3 Circular Polarization Right Hand Mode

The fields in the circular polarization right hand mode are

$$B_x = \frac{1}{2\sqrt{2}} \phi_0 k_u \cosh\left(\frac{1}{\sqrt{2}} k_u (\tilde{x} + \tilde{y})\right) \cos(k_u z) - \frac{1}{2\sqrt{2}} \phi_0 k_u \cosh\left(\frac{1}{\sqrt{2}} k_u (\tilde{x} - \tilde{y})\right) \sin(k_u z) \quad (42)$$

$$B_y = \frac{1}{2\sqrt{2}} \phi_0 k_u \cosh\left(\frac{1}{\sqrt{2}} k_u (\tilde{x} + \tilde{y})\right) \cos(k_u z) + \frac{1}{2\sqrt{2}} \phi_0 k_u \cosh\left(\frac{1}{\sqrt{2}} k_u (\tilde{x} - \tilde{y})\right) \sin(k_u z) \quad (43)$$

$$B_z = -\frac{1}{2} \phi_0 k_u \sinh\left(\frac{1}{\sqrt{2}} k_u (\tilde{x} + \tilde{y})\right) \sin(k_u z) - \frac{1}{2} \phi_0 k_u \sinh\left(\frac{1}{\sqrt{2}} k_u (\tilde{x} - \tilde{y})\right) \cos(k_u z) \quad (44)$$

Expanding to first order, the fields become

$$B_x = \frac{1}{2\sqrt{2}} \phi_0 k_u \cos(k_u z) - \frac{1}{2\sqrt{2}} \phi_0 k_u \sin(k_u z) \quad (45)$$

$$B_y = \frac{1}{2\sqrt{2}} \phi_0 k_u \cos(k_u z) + \frac{1}{2\sqrt{2}} \phi_0 k_u \sin(k_u z) \quad (46)$$

$$B_z = -\frac{1}{2} \phi_0 k_u \frac{1}{\sqrt{2}} k_u (\tilde{x} + \tilde{y}) \sin(k_u z) - \frac{1}{2} \phi_0 k_u \frac{1}{\sqrt{2}} k_u (\tilde{x} - \tilde{y}) \cos(k_u z) \quad (47)$$

These expressions can be simplified by using the following identities.

$$\cos(x) + \sin(x) = \sqrt{2} \sin\left(\frac{\pi}{4} + x\right) \quad (48)$$

$$\cos(x) - \sin(x) = \sqrt{2} \sin\left(\frac{\pi}{4} - x\right) \quad (49)$$

With these identities, the fields become

$$B_x = \frac{1}{2}\phi_0 k_u \sin\left(\frac{\pi}{4} - k_u z\right) \quad (50)$$

$$B_y = \frac{1}{2}\phi_0 k_u \sin\left(\frac{\pi}{4} + k_u z\right) \quad (51)$$

$$B_z = -\frac{1}{2}\phi_0 k_u^2 \tilde{x} \sin\left(\frac{\pi}{4} + k_u z\right) + \frac{1}{2}\phi_0 k_u^2 \tilde{y} \sin\left(\frac{\pi}{4} - k_u z\right) \quad (52)$$

Note that B_x and B_y have no x or y dependence. This means that the fields on the beam axis are the same as the measured fields.

$$B_{xb} = B_{x1} \quad (53)$$

$$B_{yb} = B_{y1} \quad (54)$$

For B_z , the field on the beam axis is obtained from the first order Taylor expansion in the transverse coordinates.

$$B_{zb} \simeq B_{z1} + \frac{\partial}{\partial x} B_z|_1 (\tilde{x}_b - \tilde{x}_1) + \frac{\partial}{\partial y} B_z|_1 (\tilde{y}_b - \tilde{y}_1) \quad (55)$$

Carrying out the derivatives, we find

$$B_{zb} = B_{z1} + \frac{1}{2}\phi_0 k_u^2 \sin\left(\frac{\pi}{4} + k_u z\right) (\tilde{x}_b - \tilde{x}_1) + \frac{1}{2}\phi_0 k_u^2 \sin\left(\frac{\pi}{4} - k_u z\right) (\tilde{y}_b - \tilde{y}_1) \quad (56)$$

$$= B_{z1} - B_{y1} k_u (\tilde{x}_b - \tilde{x}_1) + B_{x1} k_u (\tilde{y}_b - \tilde{y}_1) \quad (57)$$

In summary, we find

$$B_{xb} = B_{x1} \quad (58)$$

$$B_{yb} = B_{y1} \quad (59)$$

$$B_{zb} = B_{z1} - B_{y1} k_u (\tilde{x}_b - \tilde{x}_1) + B_{x1} k_u (\tilde{y}_b - \tilde{y}_1) \quad (60)$$

3.4 Circular Polarization Left Hand Mode

The fields in the circular polarization left hand mode are

$$B_x = \frac{1}{2\sqrt{2}}\phi_0 k_u \cosh\left(\frac{1}{\sqrt{2}}k_u(\tilde{x} + \tilde{y})\right) \cos(k_u z) + \frac{1}{2\sqrt{2}}\phi_0 k_u \cosh\left(\frac{1}{\sqrt{2}}k_u(\tilde{x} - \tilde{y})\right) \sin(k_u z) \quad (61)$$

$$B_y = \frac{1}{2\sqrt{2}}\phi_0 k_u \cosh\left(\frac{1}{\sqrt{2}}k_u(\tilde{x} + \tilde{y})\right) \cos(k_u z) - \frac{1}{2\sqrt{2}}\phi_0 k_u \cosh\left(\frac{1}{\sqrt{2}}k_u(\tilde{x} - \tilde{y})\right) \sin(k_u z) \quad (62)$$

$$B_z = -\frac{1}{2}\phi_0 k_u \sinh\left(\frac{1}{\sqrt{2}}k_u(\tilde{x} + \tilde{y})\right) \sin(k_u z) + \frac{1}{2}\phi_0 k_u \sinh\left(\frac{1}{\sqrt{2}}k_u(\tilde{x} - \tilde{y})\right) \cos(k_u z) \quad (63)$$

Expanding to first order and using the identities given above, the fields become

$$B_x = \frac{1}{2}\phi_0 k_u \sin\left(\frac{\pi}{4} + k_u z\right) \quad (64)$$

$$B_y = \frac{1}{2}\phi_0 k_u \sin\left(\frac{\pi}{4} - k_u z\right) \quad (65)$$

$$B_z = \frac{1}{2}\phi_0 k_u^2 \tilde{x} \sin\left(\frac{\pi}{4} - k_u z\right) - \frac{1}{2}\phi_0 k_u^2 \tilde{y} \sin\left(\frac{\pi}{4} + k_u z\right) \quad (66)$$

Using the methods given above for circular right hand polarization, we find for circular left hand polarization

$$B_{xb} = B_{x1} \quad (67)$$

$$B_{yb} = B_{y1} \quad (68)$$

$$B_{zb} = B_{z1} + B_{y1} k_u (\tilde{x}_b - \tilde{x}_1) - B_{x1} k_u (\tilde{y}_b - \tilde{y}_1) \quad (69)$$

4 Conclusion

The fields on the beam axis were calculated from the measured fields and the functional form of the transverse behavior of the fields. In particular, the field on the beam axis was given by the measured field plus a correction term calculated from the functional form of the fields. This procedure keeps the effect of field errors from imperfections such as magnetization direction errors of the magnet blocks which are present in the measured field.

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