Analysis of Longitudinal Space Charge Effects With Radial Dependence[*]

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Abstract

Longitudinal space charge (LSC) force can be a main effect driving the microbunching instability in the linac for an x-ray free-electron laser (FEL). In this paper, the LSC-induced beam modulation is studied using an integral equation approach that takes into account the transverse (radial) variation of LSC field for both the coasting beam limit and bunched beam. Changes of beam energy and the transverse beam size can be also incorporated. We discuss the validity of this approach and compare it with other analyses as well as numerical simulations.

INTRODUCTION

To ensure an x-ray FEL successful commission and operation, the electron beam is prepared with highest possible quality. However, such high quality electron beam is subject to various instability along the accelerator system. Due to the very small energy spread from the RF gun, the Landau damping is ineffective [1, 2, 3]. Because of the inevitable density un-smoothness of the electron beam born from the RF cathode, the space charge effect can induce large energy modulation on the beam, which leads to instability downstream [2, 4]. In this paper, we study the LSC effect taking into account acceleration, and also variation of transverse beam size during the acceleration. We study this analytically via an integral equation approach, which is compared to direct numerical simulation, and also other analytical approach [5].

COASTING BEAM THEORY

1-D formulae

Following Ref. [1], but taken into account of acceleration, the beam is described by a distribution function $f(x, x', y, y', z, \gamma; s)$ with $s = c \int \sqrt{1 - \gamma^{-2}} dt$ to be the position along the beam line. The distribution function immediately after the energy kick due to wakefield (at $\tau + 0$) is related to that immediately before (at $\tau - 0$) by

$$
\begin{aligned}
f(\mathbf{X}_\tau; \tau + 0) &= f(\mathbf{X}_\tau - \Delta \mathbf{X}; \tau - 0) \quad (1) \\
&\approx f(\mathbf{X}_\tau; \tau - 0) - \Delta\gamma \frac{\partial f(\mathbf{X}_\tau; \tau - 0)}{\partial \gamma_\tau};
\end{aligned}
$$

where $\Delta \mathbf{X} = (0, 0, 0, 0, 0, \Delta\gamma)$. Here, we focus on the longitudinal phase space only. Summing up wakefield contribution over the entire trajectory, *i.e.*, $\tau \in [0, s]$, and using the boundary notation $f[\mathbf{X}_{\tau \to s}; (\tau \to s) + 0] = f[\mathbf{X}(s); s]$,

and $f[\mathbf{X}_{\tau \to 0}; (\tau \to 0) - 0] = f_0(\mathbf{X}_0)$, the evolution of the distribution function under the influence of the wakefield is

$$
f[\mathbf{X}(s); s] = f_0(\mathbf{X}_0) - \int_0^s d\tau \frac{\partial f(\mathbf{X}_\tau; \tau - 0)}{\partial \gamma_\tau} \frac{d\gamma}{d\tau}. \quad (2)
$$

The rate of energy change due to the wakefield is

$$
\frac{d\gamma}{d\tau} = -r_e \int \frac{dk_1}{2\pi} Z(k_1; \tau) N b(k_1; \tau) e^{ik_1 z_\tau}. \quad (3)
$$

Here r_e is the electron classical radius, N is the total number of electron, γ is the Lorentz factor, and $Z(k_1; s)$ is the longitudinal impedance.

Introducing density bunching factor $b(k; s)$ as

$$
b(k; s) = \frac{1}{N} \int d\mathbf{X} e^{-ikz} f(\mathbf{X}; s), \quad (4)
$$

and using Eq. (2), we have

$$
\begin{aligned}
b[k(s); s] = b_0[k(s); s] &- \frac{ik}{N} \int d\tau \mathcal{R}_{56}(\tau \to s) \\
&\times \int d\mathbf{X}_\tau e^{-ikz(\mathbf{X}_\tau)} f(\mathbf{X}_\tau; \tau - 0) \frac{d\gamma}{d\tau}, \quad (5)
\end{aligned}
$$

where

$$
b_0(k; s) = \frac{1}{N} \int d\mathbf{X}_0 e^{-ikz} f_0(\mathbf{X}_0), \quad (6)
$$

is the bunching factor without wakefield. We have introduced the symplectic transfer matrix as $\mathbf{X}(s) \equiv \mathcal{R}(\tau \to s)\mathbf{X}(\tau)$. Now, plug Eq. (3) into Eq. (5), we have

$$
b[k(s); s] = b_0[k(s); s] + ikr_e \int d\tau \mathcal{R}_{56}(\tau \to s) \quad (7)
$$

$$
\times \int \frac{dk_1}{2\pi} Z(k_1; \tau) b(k_1; \tau) \int d\mathbf{X}_0 e^{-ikz + ik_1 z_\tau} f_0(\mathbf{X}_0),
$$

where $z_\tau = z_0 + \mathcal{R}_{56}(0 \to \tau)\Delta\gamma_0$, and $z = z_0 + \mathcal{R}_{56}(0 \to s)\Delta\gamma_0$.

Now, let us assume that

$$
f_0(\mathbf{X}_0) = \bar{f}_0(\mathbf{X}_0) + \hat{f}_0(\mathbf{X}_0), \quad (8)
$$

where $\bar{f}_0(\mathbf{X}_0)$ is the average distribution function, and $\hat{f}_0(\mathbf{X}_0)$ is the initial microbunching. For microbunching wavelength much smaller than the bunch length, we could assume uniform longitudinal distribution, hence coasting beam, in z, and Gaussian in $\Delta\gamma$ for the average distribution function, *i.e.*, we assume

$$
\bar{f}_0(\mathbf{X}_0) = \frac{n_0}{\sqrt{2\pi}\sigma_{\Delta\gamma}} \exp\left\{-\frac{\Delta\gamma^2}{2\sigma_{\Delta\gamma}^2}\right\}, \quad (9)
$$

where n_0 is the average line density. Within the linear theory, we could neglect the $\hat{f}_0(\mathbf{X}_0)$ in completing the integral in Eq. (7). In doing so, we get

$$b[k(s);s] = b_0[k(s);s] + \int_0^s d\tau K(\tau,s)b[k(\tau);\tau], \quad (10)$$

with the kernel of the integral equation as

$$\begin{aligned} K(\tau,s) &= ik(s)\mathcal{R}_{56}(\tau \to s)\frac{I(\tau)Z[k(\tau);\tau]}{I_A} \\ &\times \exp\left\{-k_0^2\mathcal{R}_{56}^2(\tau \to s)\sigma_{\Delta\gamma}^2/2\right\}, \quad (11) \end{aligned}$$

where

$$\mathcal{R}_{56}(\tau \to s) \equiv \int_\tau^s \frac{dx}{\gamma(x)^3[1-\gamma(x)^{-2}]}. \quad (12)$$

It is worth noticing that due to the uniform distribution in z, there is only a single frequency selected in k-integral in Eq. (7). Should we work on a Gaussian distribution in z, we will deal with a multi-frequency theory.

According to Eq. (3), the resulting accumulated energy modulation spectrum is then

$$\begin{aligned} \Delta\gamma[k(s);s] &= -\int_0^s d\tau \frac{I_0 Z[k(\tau);\tau]b[k(\tau);\tau]}{I_A} \\ &\times \exp\left\{-k^2\mathcal{R}_{56}^2(\tau \to s)\sigma_{\Delta\gamma}^2/2\right\}, \quad (13) \end{aligned}$$

where $I_0 = ecn_0$ is the peak current with $n_0 = N/L$ the peak density, and $I_A \approx 17045$ Amp is the Alfvén current.

Radial Dependance

If the transverse dynamics and the longitudinal dynamics are separable, *i.e.*, we assume that the distribution function is factorable

$$f(\mathbf{X};s) = f_r(r;s)f_z(z;s), \quad (14)$$

with the normalization of

$$\int d\mathbf{r}\, f_r(r;s) = 1, \quad (15)$$

then the three dimensional problem can be simplified into one dimensional problem.

Transverse averaging approach One approach is to average out the transverse variables. The energy change rate is then

$$\begin{aligned} \frac{d\gamma(r,z;s)}{ds} &= -r_e \int dz' dr'\, w\,(z-z',r,r')\, f\,(z',r';s) \\ &= -r_e \int dr' \int \frac{dk}{2\pi} Z\,(k;r,r')\, e^{ikz} \\ &\times \int dz' e^{-ikz'} f_r\,(r';s)\, f_z\,(z';s) \\ &= -r_e \int \frac{dk}{2\pi} \bar{Z}[k(s);r,s]Nb[k(s);s]e^{ikz}, \quad (16) \end{aligned}$$

where we have introduced the wakefield as

$$w(z,r,r',s) = \int \frac{dk}{2\pi} Z(k;r,r',s)e^{ikz}; \quad (17)$$

and we have also defined the averaged impedance as

$$\bar{Z}(k;r,s) = \int d\mathbf{r}' Z(k;r,r',s)f_r(r';s). \quad (18)$$

The bunching factor could be simplified as

$$b(k;s)=\frac{1}{N}\int d\mathbf{X}e^{-ikz}f(\mathbf{X};s)=\frac{1}{N}\int dz e^{-ikz}f_z(z;s), \quad (19)$$

according to Eqs. (14) and (15).

Now, plug Eq. (16) into Eq. (5), we have

$$\begin{aligned} b[k(s);s] &= b_0[k(s);s] + ikr_e \int d\tau \mathcal{R}_{56}(\tau \to s)\int d\mathbf{X}_0 \quad (20) \\ &\times \int \frac{dk_1}{2\pi} \bar{Z}(k_1;r,\tau)b(k_1;\tau)e^{-ikz+ik_1 z_\tau} f_0(\mathbf{X}_0). \end{aligned}$$

We then do the linearization and complete the integrals as we did for the 1-D case. In doing so, we formally get the same equation for the evolution of the bunching factor as in Eq. (10). However, the kernel of the integral equation is different from that given in Eq. (11). Here, the kernel is

$$\begin{aligned} K(\tau,s) &= ik(s)\mathcal{R}_{56}(\tau \to s)\frac{I(\tau)\bar{\bar{Z}}[k(\tau);\tau]}{I_A} \\ &\times \exp\left\{-k_0^2\mathcal{R}_{56}^2(\tau \to s)\sigma_{\Delta\gamma}^2/2\right\}, \quad (21) \end{aligned}$$

where the double-averaged impedance is defined as

$$\bar{\bar{Z}}(k;s) = \int d\mathbf{r}\, \bar{Z}(k;r,s)f_r(r;s). \quad (22)$$

According to Eq. (16), the resulting accumulated energy modulation spectrum is then

$$\begin{aligned} \Delta\gamma[k(s);s] &= -\int_0^s d\tau \frac{I_0 \bar{\bar{Z}}[k(\tau);\tau]b[k(\tau);\tau]}{I_A} \\ &\times \exp\left\{-k^2\mathcal{R}_{56}^2(\tau \to s)\sigma_{\Delta\gamma}^2/2\right\}. \quad (23) \end{aligned}$$

Radial variable as parameter Should we not do the average in Eq. (18), we can keep the r-dependence. This approach was recently taken in Ref. [5, 6]. In our approach, we introduce a radial-dependent bunching factor

$$\begin{aligned} b(k;s,r) &= \frac{1}{N}\int dz e^{-ikz}f(\mathbf{r},z;s) \\ &= f_r(r;s)\frac{1}{N}\int dz e^{-ikz}f_z(z;s), \quad (24) \end{aligned}$$

so that

$$b(k;s) = \frac{\iint_{\Sigma_\perp} d\mathbf{r}\, b(k;s,r)}{\Sigma_\perp}. \quad (25)$$

The energy change rate is then

$$
\begin{aligned}
\frac{d\gamma(r,z;s)}{ds} &= -r_e \int dz' d\mathbf{r}' w\left(z - z', r, r'\right) f\left(z', r'; s\right) \\
&= -r_e \int d\mathbf{r}' \int \frac{dk}{2\pi} Z\left(k; r, r'\right) e^{ikz} \\
&\quad \times \int dz' e^{-ikz'} f_r\left(r'; s\right) f_z\left(z'; s\right) \quad (26) \\
&= -r_e \int d\mathbf{r}' \int \frac{dk}{2\pi} Z[k(s); r, r'] N b[k(s); s, r'] e^{ikz},
\end{aligned}
$$

Therefore the bunching factor evolves as

$$
\begin{aligned}
b[k(s); s, r] &= b_0[k(s); s, r] \quad (27) \\
&+ \int_0^s d\tau \int d\mathbf{r}' K(\tau, s, r, r') b[k(\tau); \tau, r']
\end{aligned}
$$

with

$$
K(\tau, s, r, r') = ik(s)\mathcal{R}_{56}(\tau \to s)\frac{I(\tau)Z[k(\tau); \tau, r, r']}{I_A \Sigma_\perp}. \quad (28)
$$

The corresponding evolution for the energy modulation is

$$
\Delta\gamma(s,r) = -\int_0^s d\tau \int d\mathbf{r}' \frac{I_0 Z[k(\tau); \tau, r, r'] b[k(\tau); \tau, r']}{I_A \Sigma_\perp}. \quad (29)
$$

Hence, the average energy modulation is

$$
\Delta\gamma(s) = \frac{\iint_{\Sigma_\perp} d\mathbf{r} \Delta\gamma(s,r)}{\Sigma_\perp}. \quad (30)
$$

BUNCHED BEAM THEORY

In reality, the electron beam's longitudinal distribution is not uniform, hence let us now improve the theory to deal with bunched beam, and so multi frequency case.

1-D formulae

For 1-D theory, the derivation up to Eq. (7) stays the same. For a Gaussian longitudinal distribution, we assume

$$
\bar{f}_0(\mathbf{X}_0) = \frac{N}{2\pi\sigma_{\Delta\gamma}\sigma_z} \exp\left\{ -\frac{\Delta\gamma^2}{2\sigma_{\Delta\gamma}^2} - \frac{z^2}{2\sigma_z^2} \right\}. \quad (31)
$$

Here we use the same notation for $\bar{f}_0(\mathbf{X}_0)$ as in Eq. (9) without worrying about any possible confusion.

Completing the integral in Eq. (7), we obtain the evolution for the bunching factor as

$$
\begin{aligned}
b[k(s); s] &= b_0[k(s); s] \quad (32) \\
&+ \int_0^s d\tau \int \frac{dk(\tau)}{2\pi} K[k(\tau), k(s); \tau, s] b[k(\tau); \tau],
\end{aligned}
$$

with the integral kernel to be

$$
\begin{aligned}
K[k(\tau), k(s); \tau, s] &= ik(s)\mathcal{R}_{56}(\tau \to s)\frac{I(\tau)Z[k(\tau); \tau]}{I_A} \\
&\times \exp\left\{ -\frac{[k(s) - k(\tau)]^2 \sigma_z^2}{2} \quad (33) \right. \\
&\left. -\frac{[k(s)\mathcal{R}_{56}(s) - k(\tau)\mathcal{R}_{56}(\tau)]^2 \sigma_{\Delta\gamma}^2}{2} \right\}.
\end{aligned}
$$

The corresponding accumulated energy modulation spectrum is then

$$
\begin{aligned}
\Delta\gamma[k(s); s] &= -\int_0^s d\tau \int \frac{dk(\tau)}{2\pi} \frac{I_0 Z[k(\tau); \tau] b[k(\tau); \tau]}{I_A} \\
&\times \exp\left\{ -\frac{[k(s) - k(\tau)]^2 \sigma_z^2}{2} \quad (34) \right. \\
&\left. -\frac{[k(s)\mathcal{R}_{56}(s) - k(\tau)\mathcal{R}_{56}(\tau)]^2 \sigma_{\Delta\gamma}^2}{2} \right\}.
\end{aligned}
$$

Radial dependance

Let us take the radial dependance into consideration. With the transverse averaging approach, we simply replace $Z[k(\tau); \tau]$ in Eqs. (32), (33), and (34) by $\bar{\bar{Z}}[k(\tau); \tau]$ defined in Eq. (22).
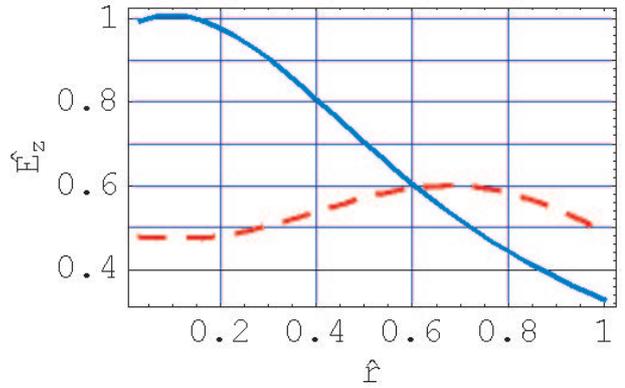

Figure 1: Example of Figs. 10 and 11 of Ref. [5]. The initial density modulation is $\hat{a}_{1d} = e^{-\hat{r}^2/(2\sigma)}$ with $\sigma = 0.1$, and $q = 1$. The solid curve (blue) is at $\hat{z} = 0$, and the dashed curve (red) is at $\hat{z} = 10$. Notations have same meaning as in Ref. [5], the $\hat{E}_z$ is normalized.

LSC IMPEDANCE WITH RADIAL DEPENDENCE

Having setup the frame work above, let us now find the LSC impedance with radial dependence.

Green function for a $\delta-$ring

What we need is the Green function for a $\delta-$ring. We omit the derivation here, for a $\delta-$ring at r', the LSC impedance is

$$
\begin{aligned}
Z(r, r', z) &= \frac{k}{\gamma}\left[\Theta(r' - r)\frac{2}{\gamma}K_0\left(\frac{kr'}{\gamma}\right)I_0\left(\frac{kr}{\gamma}\right) \right. \\
&+ \Theta(r - r')\frac{1}{kr'}K_0\left(\frac{kr}{\gamma}\right)\left\{ 2I_1\left(\frac{kr'}{\gamma}\right) \right. \\
&\left.\left. + \frac{kr'}{\gamma}\left[I_0\left(\frac{kr'}{\gamma}\right) + I_2\left(\frac{kr'}{\gamma}\right) \right] \right\} \right]. \quad (35)
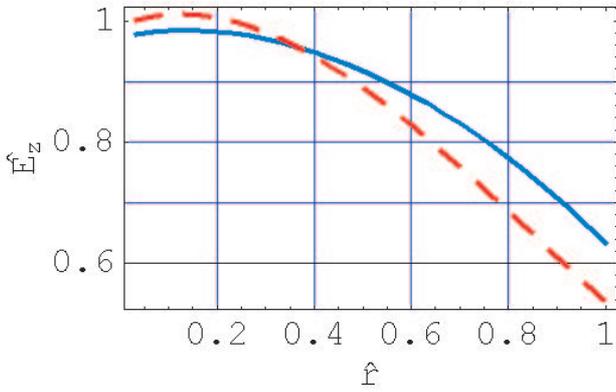\end{aligned}
$$

Figure 2: Example of Figs. 14 and 15 of Ref. [5]. The initial density modulation is $\hat{a}_{1d} = 1$ with $\sigma = 0.1$, and $q = 1$. The solid curve (blue) is at $\hat{z} = 0$, and the dashed curve (red) is at $\hat{z} = 10$. Notations have same meaning as in Ref. [5], the $\hat{E}_z$ is normalized.

For parabolic distribution, one can get a closed form for the average impedance defined in Eq. (18).

COMPARISON

Let us now compare the results to some other analytical approach [5], and also numerical simulation.

Comparison with other analytical approach

Compared to theory in Refs. [5, 6], our approach has the advantage of treating the real beam line, where beam energy and transverse beam size are varying, and the beam is bunched. Nevertheless, let us make some comparison with Ref. [5]. In their paper, they introduce a dimensionless parameter $q = k_m r_0/\gamma_z$, with $k_m = 2\pi/\lambda_m$ where λ_m is the modulation wavelength; r_0 is typical transverse beam size; and γ_z is the longitudinal Lorentz factor. Their theory reduces to 1-D formula when $q \to \infty$. Hence, let us compare for the case of having a small $q = 1$. In Figs. 1 and 2, we study the same examples in Figs. 10, 11, 14, and 15 of Ref. [5]. The results are almost the same.

Comparison with PARMELA

Having compared with the results in Ref. [5] for their limited applicability, let us now deal with realistic beam line. The example we study is for beam with energy $E = 5.7$ MeV and peak current of 100 A. We study a 3 m long drift space, with betatron focusing, hence the rms transverse beam size σ_r is varying from 0.5 mm to 3.7 mm. The initial density modulation is 5 % with wavelength of 0.5 mm. This yields that $q \in [1.0, 7.2]$ taking $r_0 = \sqrt{3}\sigma_r$. We show in Fig. 3 the results of the four approaches developed in this paper. As a comparison, PARMELA simulation [7] for a bunched beam, with rms bunch length $\sigma_z = 0.83$ mm, is also presented.
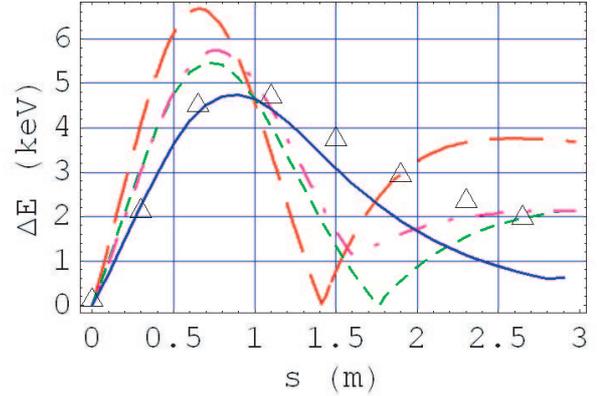

Figure 3: Comparison of four different analytical approaches developed in this paper with PARMELA simulation. The long-dashed curve (red) is the 1-D coasting beam approach, the dashed curve (green) is the coasting beam with radial dependence approach, the dash-dotted curve (purple) is the 1-D bunched beam approach, the solid curve (blue) is the bunched beam approach with radial dependence, and the "△" (black) is the PARMELA simulation.

DISCUSSION

As shown in Fig. 3, it is clear that the coasting beam theory is over simplified in dealing with LSC in real situation where the beam is bunched. Even with radial dependence, the coasting beam theory [5] can not capture some of the features in simulations for bunched beam, which is the realistic situation. It is worthwhile to point out that in Ref. [5, 6], the $r-$dependance comes in also as a parameter, *i.e.*, $f(\mathbf{X}; s) = f_r(r)f_z(z; s)$, but not $f_r(r; s)$ as in Eq. (14) of our paper. For Ref. [3], since we were dealing with beam having energy higher than 135 MeV, the density microbunching is mostly frozen; hence no dynamics as discussed here. As we find, based on a bunched beam theory, and further taking into account the radial dependence, the analytical approach developed here show a good agreement with the PARMELA results. The authors would like to thank C. Limborg of SLAC and M. Borland of ANL for many stimulating discussions.

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